Techniques for carefully sketching functions

When sketching a graph of a function $f(x)$, you want to clearly indicate all the important features of the function, including: its domain, the x- and y-intercepts (maybe), intervals on which the function is increasing and decreasing, local maxima and minima, intervals where the function is concave up and concave down, inflection points, horizontal asymptotes and vertical asymptotes.

A **Determine the domain.** Are there any places where the function is undefined? For example, is there a denominator which could be zero? Does your function include functions such as $\sqrt{x}$ or $\ln(x)$, which are not defined for negative x-values?

B **Find interesting points** Determine the y-intercept. Determine the x-intercepts if it's not too hard. To determine the y-intercept, evaluate $f(0)$. To determine the x-intercept, solve $f(x) = 0$. (This may be hard, so you may approximate or skip when appropriate. In addition, not every function has an x- or y-intercept.) Note any points where the function is not defined.

C **Symmetry** Note any symmetry. Is the function even like x^2 and $\cos(x)$? Is it odd like x^3 and $\sin(x)$? Is it periodic? Even functions satisfy $f(-x) = f(x)$ and odd functions satisfy $f(-x) = -f(x)$. Periodic functions satisfy $f(x) = f(x + P)$ for some period P.

D **Asymptotic behaviour** We need to determine the behavior of the function at $\pm\infty$ and near any points where the function is not defined.

A horizontal asymptote is a horizontal line which the function approaches "eventually"; that is, $y = L$ is a horizontal asymptote if $\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$. So you need to compute both those limits to determine the behavior of the function as x increases without bound.

Suppose the function is not defined at $x = a$. Then we need to compute $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ to determine whether f has a vertical asymptote at a and what the behavior of the function is near the asymptote.

E **Determine intervals of increase, decrease, and local maxima/minima** To do so, solve $f'(x) = 0$ for x to find the critical points, and then make a first derivative chart to determine the behavior of the function in the intervals determined by the critical points. Finally, by considering where the function changes from increasing to decreasing (or vice versa), you can determine the local maxima and minima.

F **Determine intervals of concave up, concave down, and inflection points** To do so, solve $f''(x) = 0$ for x to find the potential inflection points, and then make a second derivative chart to determine the behavior of the function in the intervals determined by the potential inflection points. Finally, by considering where the function changes from CU to CD (or vice versa), you can determine the inflection points.

G Finally, stitch together all of the information you have determined about the function and sketch the graph of the function, indicating all important features.

Instructions: Work in groups of two or three. I suggest you discuss what work needs to be done, but partition out the work so that not everyone is doing every computation.

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1 Warm up: Sketch a function

Sketch a graph of a function that has **all** of the following properties. Indicate where the function is increasing and decreasing, any local maxima and minima, intervals where the function is concave up and concave down, any inflection points, any horizontal asymptotes and vertical asymptotes

- $\lim_{x \to 4} f(x) = -\infty$
- $\lim_{x \to \infty} f(x) = 0$
- $f(-3) = 0$
- $f'(-5) = f'(1) = f'(7) = 0$
- $f'(x) < 0$ on $(-\infty, -5), (1, 4), (7, \infty)$
- $f'(x) > 0$ on $(-5, 1), (4, 7)$
- $f''(x) < 0$ on $(-3, 4), (4, 8)$
- $f''(x) > 0$ on $(-\infty, -3), (8, \infty)$

2 Sketch some functions

Sketch careful graphs of **any three** of the following functions, indicating all the important features of the function, including for each function: its domain, the x- and y-intercepts, intervals on which the function is increasing and decreasing, local maxima and minima, intervals where the function is concave up and concave down, inflection points, horizontal asymptotes and vertical asymptotes.

(a) $f(x) = xe^{-(x^2)}$

(b) $g(x) = 8x^2 - x^4$

(c) $h(x) = x\sqrt{4-x}$

(d) $k(x) = \dfrac{2x}{1-x^2}$

(e) $j(x) = \dfrac{\ln(x)}{x}$

You may find the following table of (simplified) derivatives to be helpful:

function	first derivative	second derivative
$f(x)$	$(1-2x^2)e^{-x^2}$	$4x(x^2-\frac{3}{4})e^{-x^2}$
$g(x)$	$16x - 4x^3$	$16 - 12x^2$
$h(x)$	$\dfrac{8-3x}{2\sqrt{4-x}}$	$\dfrac{3x-16}{4(4-x)^{3/2}}$
$k(x)$	$2\dfrac{x^2+1}{(x^2-1)^2}$	$-4\dfrac{x(x^2+3)}{(x^2-1)^3}$
$j(x)$	$\dfrac{1-\ln(x)}{x^2}$	$\dfrac{-3+2\ln(x)}{x^3}$