1. Find the solution of

$$y' + 4y = f(t); \qquad y(0) = 7$$

where

$$f(t) = \begin{cases} 0 & t < 1 \\ e^{-2t} & t > 1. \end{cases}$$

Solution:

Notice that

$$\begin{aligned} f(t) &= u_1(t)e^{-2t} \\ &= u_1(t)e^{-2(t-1)-2} \\ &= e^{-2}u_1(t)e^{-2(t-1)}. \end{aligned}$$

By our switching rule,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\left\{e^{-2}u_1(t)e^{-2(t-1)}\right\} \\ &= e^{-2}\mathcal{L}\left\{u_1(t)e^{-2(t-1)}\right\} \\ &= e^{-2}\mathcal{L}\left\{\mathcal{S}_1\left\{e^{-2t}\right\}\right\} \\ &= e^{-2}e^{-s}\mathcal{L}\left\{e^{-2t}\right\} \\ &= e^{-2}e^{-s}\frac{1}{s+2}. \end{aligned}$$

Taking the Laplace transform of both sides of the differential equation,

$$(s+4)Y - y(0) = e^{-2}e^{-s}\frac{1}{s+2}$$

and therefore

$$Y = \frac{7}{s+4} + e^{-2}e^{-s}\frac{1}{(s+2)(s+4)}.$$

Using partial fractions,

$$\frac{1}{(s+2)(s+4)} = \frac{1}{2}\left[\frac{1}{s+2} - \frac{1}{s+4}\right].$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s+4)}\right\} = \frac{1}{2}\left[e^{-2t} - e^{-4t}\right].$$

By our rule for switching,

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+2)(s+4)}\right\} &= \mathcal{S}_1\left\{\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s+4)}\right\}\right\} \\ &= \mathcal{S}_1\left\{\frac{1}{2}\left[e^{-2t} - e^{-4t}\right]\right\} \\ &= u_1(t)\frac{1}{2}\left[e^{-2(t-1)} - e^{-4(t-1)}\right]. \end{aligned}$$

Hence

$$\begin{aligned}
y &= \mathcal{L}^{-1}\left\{\frac{7}{s+4} + e^{-2}e^{-s}\frac{1}{(s+2)(s+4)}\right\} \\
&= 7\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + e^{-2}\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{(s+2)(s+4)}\right\} \\
&= 7e^{-4t} + e^{-2}u_1(t)\frac{1}{2}\left[e^{-2(t-1)} - e^{-4(t-1)}\right] \\
&= 7e^{-4t} + u_1(t)\frac{1}{2}\left[e^{-2t} - e^{-4t+2}\right].
\end{aligned}$$