1. Text 2.5. (Again) Present a (correct) triangulation, but **leave it to the reader to verify that it is a triangulation**. The reader will let you know if it is not correct. ⌣̈ Prove that is is minimal by a solely combinatorial argument; see the hint in the book. Before using the hint, you must justify it (briefly).

Solution:
Let V, E, and T be the number of vertices, edges, and triangles in a triangulation of a surface. We have the following combinatorial arguments: each triangle has 3 edges and each edge is contained in exactly two triangles. Hence

$$3T = 2E.$$

Also, each edge joins two distinct vertices. There are at most $\binom{V}{2} = V(V-1)/2$ ways to do this, so

$$2E \le V(V-1).$$

For the projective plane, $\chi = 1$ and hence

$$V - E + T = 1.$$

Using the relationship between E and T we have

$$V - \frac{1}{3}E = 1$$

and therefore

$$2E = 6(V-1).$$

Our estimate for $2E$ in terms of the number of vertices then yields

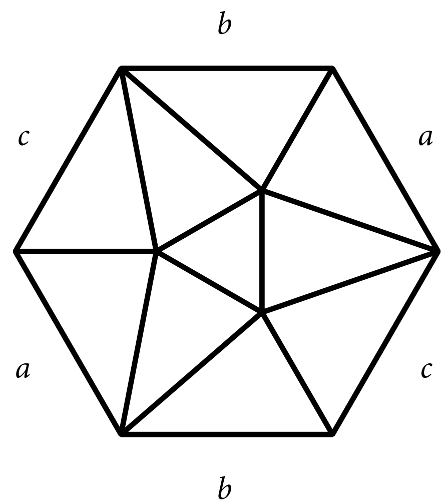
$$6(V-1) \le V(V-1).$$

Since $V-1$ is positive, we conclude that

$$V \ge 6.$$

Thus the minimum number of vertices is 6. For such a triangulation we must have $E = 3(V-1)$ and $T = \frac{2}{3}E = 2(V-1)$. That is, once the number of vertices is known, so are the numbers of edges and triangles. To minimize the number of simplices, it suffices to minimize the number of vertices, and 6 is the minimum possible.

Here is a triangulation with 6 vertices:



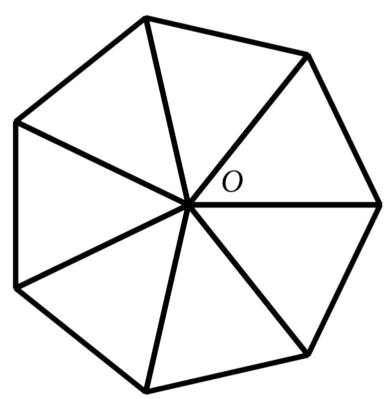
2. (Euler characteristic of planar models, take 2) Let P be a standard n-gon (with $n \geq 3$) (i.e. the polygon with vertices $\exp(i\theta_k)$ with $\theta_k = k/n$ for $k = 0, 1, \ldots, n-1$). Suppose we have a gluing scheme for the edges where each edge is glued to at most one other edge by means of an affine map. Let Q be the resulting quotient space, and let n_v and n_e be the number of vertices and edges respectively after identification.

a) Show by direct construction that Q can be triangulated.

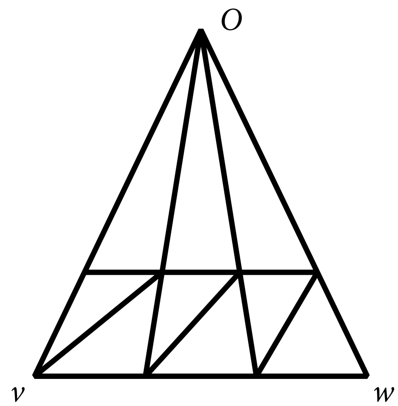
b) Show that $\chi(Q) = n_v - n_e + 1$.

c) Compute the Euler characteristic of the torus, the Klein bottle, a sphere with 3 holes, the Möbius band, and a torus with a hole.

Solution, part a:
We construct an initial triangulation by adding the center point to the polygon and adding spokes as follows.



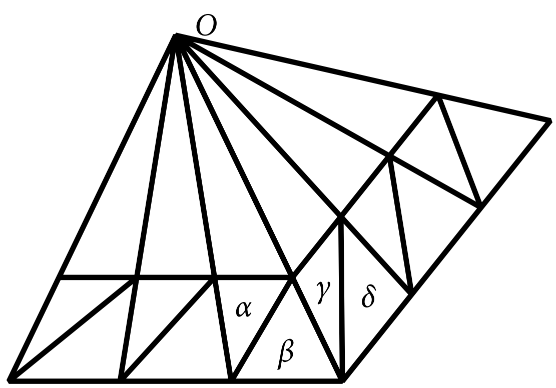
Now focusing on one of the triangles (which we will call a master triangle), we further triangulate it as follows:



This evidently gives a triangulation of the n-gon. Each pair of adjacent triangles intersects in a common face. Each triangle has at most one edge or one vertex adjacent to the outer boundary where gluing occurs. So after gluing, a pair of triangles can pick up at most one additional vertex or face of intersection. Thus our triangulation will fail only if there are two triangles that are currently adjacent and that become adjacent on a distinct face after gluing.

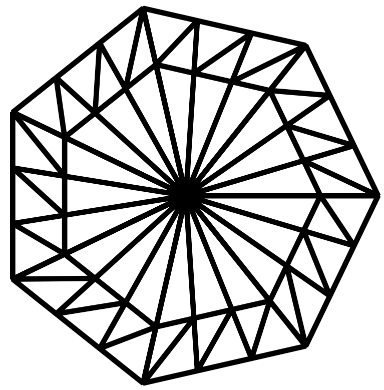
Within a single master triangle, the only possible new identifications occur at the vertices v and w. No triangle incident with v is incident with w before gluing, so no problems can occur here.

Now consider master triangles T and T' that meet only at the origin O. Then any triangle from T incident with the boundary is not incident with any triangle from T'. So again, no problems can occur here.



Thus it remains to consider the case of triangles in two adjacent master triangles. We need only consider pairs of triangles from different master triangles that are adjacent to the boundary. By symmetry, it then suffices to consider the pairs (α, γ), (β, γ), and (β, δ). The intersection of α with the boundary is never identified with the intersection of γ with the boundary, so no new intersections occur after gluing. The intersection of β and γ is their common edge regardless of whether any gluing occurs. The intersection of β and δ is just their common vertex, unless the boundaries of the two master triangles are glued with opposite orientations, in which case the intersection of β and δ is their boundary edge. In both cases, the intersection is a common face.

Hence we have a triangulation of Q.



Solution, part b:
The triangulation of the n-gon has $6n+1$ vertices, $15n$ edges, and $9n$ faces. After gluing, we lose $3n$ boundary vertices and n boundary edges, and replace them with n_v vertices and n_e edges. Thus

$$\chi(K) = [6n+1-3n+n_v] - [15n-3n+n_e] + [9n] = 3n+1+n_v-12n-n_e+9n = n_v-n_e+1.$$

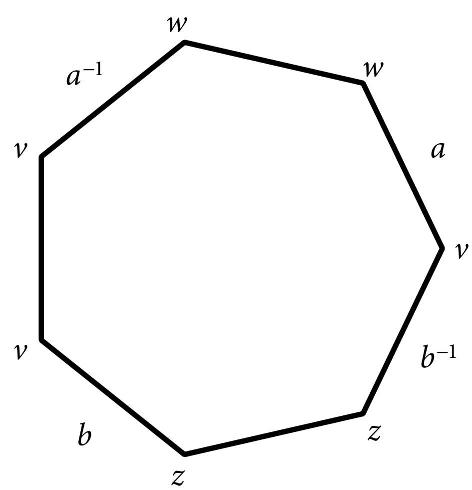
Solution, part c:
Consider the standard planar model of the torus. There are two boundary edges and a single boundary vertex. Hence

$$\chi(T^2) = 1-2+1 = 0.$$

Consider the standard planar model of the Klein bottle. There are again two boundary edges and a single boundary vertex. Hence

$$\chi(K) = 1-2+1 = 0.$$

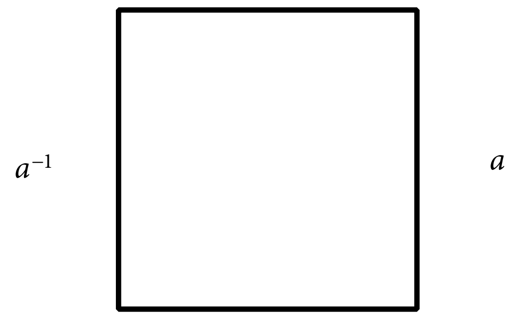
Consider the following planar model for a sphere with three holes:



There are three vertices (v, w, and z) and 5 edges (a, b, and three that are not glued). Hence

$$\chi = 3-5+1 = -1$$

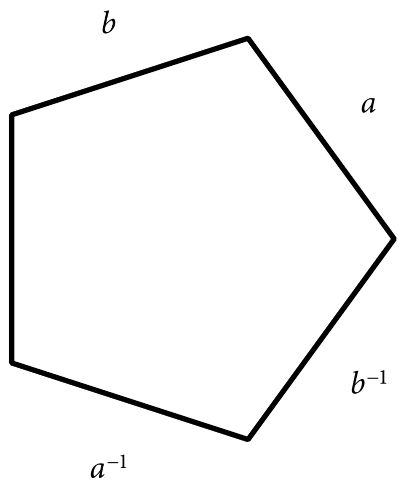
Consider the following planar model for a Möbius band:



There are two vertices (one at each end of a) and 3 edges (a and two that are not glued). Hence

$$\chi = 2 - 3 + 1 = 0$$

Consider the following planar model for a torus with a hole:



There is a single vertex and 3 edges (a, b, and one that is not glued). Hence

$$\chi = 1 - 3 + 1 = -1$$

3. Identify the following surface: $\langle a, b, c : abacb^{-1}c^{-1}\rangle$.

a) Compute its Euler characteristic. (Use problem 2!)

b) Use cut/paste maneuvers to exhibit it as a connected sum of projective planes. Express your operations in terms of surface presentations.

Solution, part a:
One readily verifies that there is exactly one vertex class in this presentation. Hence the Euler characteristic for this presentation on a hexagon is

$$\chi = 1 - 3 + 1 = -1.$$

This must be a connected sum of three projective planes.

Solution, part b:
We have the following sequence of presentations (ommitting the notation for the sets):

$$
\begin{aligned}
\langle abacb^{-1}c^{-1}\rangle &\approx \langle abe^{-1}, eacb^{-1}c^{-1}\rangle && \text{(cut)}\\
&\approx \langle eb^{-1}a^{-1}, acb^{-1}c^{-1}e\rangle && \text{(reflect and rotate)}\\
&\approx \langle eb^{-1}cb^{-1}c^{-1}e\rangle && \text{(paste)}\\
&\approx \langle b^{-1}cb^{-1}c^{-1}ee\rangle && \text{(rotate)}\\
&\approx \langle b^{-1}cf, f^{-1}b^{-1}c^{-1}ee\rangle && \text{(cut)}\\
&\approx \langle f^{-1}c^{-1}b, b^{-1}c^{-1}eef^{-1}\rangle && \text{(reflect and rotate)}\\
&\approx \langle f^{-1}c^{-1}c^{-1}eef^{-1}\rangle && \text{(paste)}\\
&\approx \langle f^{-1}f^{-1}c^{-1}c^{-1}ee\rangle && \text{(rotate).}
\end{aligned}
$$

This last presentation is the presentation of the connected sum of three projective planes.