1. In this problem, we will seek a solution to the differential equation

$$\begin{aligned} f'(t) &= F(t, f(t)) \\ f(0) &= a \end{aligned} \tag{1}$$

where a is a fixed constant.

Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous and that it is Lipschitz in its second variable. That is, there exists a constant K such that

$$|F(x, y_1) - F(x, y_2)| \le K|y_1 - y_2|$$

for all $x \in \mathbb{R}$ and all $y_1, y_2 \in \mathbb{R}$.

Define $G : C[0, T] \to C[0, T]$ by

$$G(f)(t) = a + \int_0^t F(s, f(s))\, ds.$$

1. Explain why $G(f) \in C[0, T]$ if $f \in C[0, T]$.
2. Prove that if there exists a solution f of the differential equation, then it is a fixed point of G. That is, $G(f) = f$.
3. Show that G is a Lipschitz function with constant TK, where K is the Lipschitz constant for F.
4. Prove that if $K < 1/T$, then there exists a fixed point for G. You may assume the completeness of $C[0, T]$.
5. Prove that if $K < 1/T$, then there exists a solution of the ODE (1).

2. Carothers 8.3

3. Carothers 8.12

4. Carothers 8.17

5. Carothers 8.28

6. Carothers 8.29

7. Carothers 8.38

8. Carothers 8.40

9. Carothers 8.48

10. Carothers 8.66

11. Carothers 8.67

12. Let X be a compact metric space, and let $\mathcal{X}$ be the set of non-empty closed subsets of X. If $a \in X$ and $B \in X$, we define $d(a, B) = \inf_{b \in B} d(a, b)$. We define a metric, called the Hausdorff distance, on $\mathcal{X}$ by

$$H(A, B) = \sup_{a \in A} d(a, B) + \sup_{b \in B} d(b, A).$$

a) Suppose $X \subseteq \mathbb{R}^2$ is the closed ball ball of radius 100, A is the closed square with side length 1 centered at the origin, and B is the closed ball of radius 1/4 centered at the point (1/2, 1/2). Draw a picture of the arrangement and compute $H(A, B)$. (No rigor here please!)

b) Show that H is a metric on X.