Recall last class that we found a class of functions that map the upper half complex plane to itself, and, when restricted to the boundary (i.e. the real axis) are fractional linear transformations. These are the functions of the form

$$f(z) = \frac{az + b}{cz + d} \qquad g(z) = \frac{a\bar{z} + b}{c\bar{z} + d}$$

where a, b, c, and d are all real numbers. We will call these functions Möbius transformations. The group of Möbius transformations is generated by the translations $h_d(z) = z + d$, scalings $f_k(z) = kz$, the reflection $n(z) = -\bar{z}$, and circular inversion $g(z) = -1/\bar{z}$.

1. What is the formula in terms of z and $\bar{z}$ for the vertical line passing through the point $x = a$ on the real axis?

2. What is the formula in terms of z and $\bar{z}$ for the unit circle?

3. What is the formula in terms of z and $\bar{z}$ for the circle of radius R centered at $x = a$?

We will say a set $\mathcal{L}$ is a hyperbolic line if it is either a vertical line or a the top half of a circle centered on the real axis.

4. Show that given two distinct points in the upper half plane that there is a unique hyperbolic line passing between them.

5. Consider a vertical line. Show that the image of this line under the map $z \to -1/\bar{z}$ is a hyperbolic line.

6. Consider a circle centered on the real axis. Show that the image of this hyperbolic line under the map $z \to -1/\bar{z}$ is again a hyperbolic line.

7. Show that the image of a hyperbolic line under a Möbius transformation is again a hyperbolic line.

8. Find a Möbius transformation that takes the unit circle $z\bar{z} = 1$ to the vertical line $x = 0$.

9. What is the image of the vertical lines $x = 2$, $x = 3$, and the circle of radius $1/2$ centered at $x = 5/2$ under the map you found in the previous problem?