

Lecture Notes on Surfaces

Math 412 (Spring 2005)

1. Surfaces

Intuitively, a surface is a subset of \mathbb{R}^3 that “looks like” a plane when you zoom in on it, in the same way that the graph of a smooth function looks like a line when you zoom in on it. Surfaces are more complex objects than curves, and to appropriately describe them we will require a more complex definition than we had for curves. In this section we will gradually build the definition of a surface so as to understand the meaning of each of its requirements. We start with a definition that generalizes the notion of a curve.

Definition 1.1 A (smooth) patch is a smooth function $\sigma : U \rightarrow \mathbb{R}^3$, where U is an open subset of \mathbb{R}^2 . We say σ is **regular** if the Jacobian matrix of σ has rank 2 at every point of U . The **trace** or **image** of σ is the set $\sigma(U)$.

If $\sigma(u, v) = (x(u, v), y(u, v), z(u, v))$, then the Jacobian matrix of σ is

$$D\sigma = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}.$$

We also use the notation $\sigma_u = \frac{\partial}{\partial u}\sigma$ and $\sigma_v = \frac{\partial}{\partial v}\sigma$, so that $D\sigma = [\sigma_u | \sigma_v]$. The Jacobian matrix $D\sigma$ has rank 2 if and only if σ_u and σ_v are linearly independent. As we will see, this condition helps ensure that the trace of σ looks like a plane at small scales. For example, the function $\sigma(u, v) = (u, 0, 0)$ is a non-regular patch since its Jacobian matrix is

$$D\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The trace of σ is the x axis, which clearly does not look like a plane at any scale.

Exercise 1.1 Let $\sigma : U \rightarrow \mathbb{R}^3$ be a patch, and let $q \in U$. Show that the following are equivalent.

1. $D\sigma_q$ has rank 2.
2. $\sigma_u(q) \wedge \sigma_v(q) \neq 0$.
3. $D\sigma_q$ has a 2×2 minor with non-zero determinant.

A first attempt to define a surface might be to call a surface the trace of a regular patch. But this approach leads to several difficulties. First, if the patch is not injective, there might be

places where the trace does not look like a plane at small scales. For example, consider the regular patch $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$\sigma(u, v) = (\sin(u), \sin(2u), v).$$

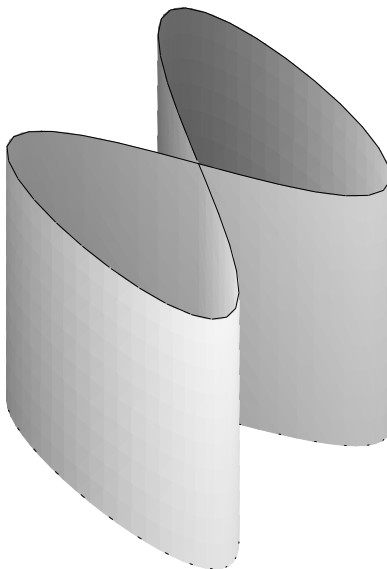


Figure 1: The trace of σ .

The trace of σ is a cylinder with figure-eight shaped cross-sections. The origin belongs to the trace of σ , but it seems clear that the trace of σ near to origin does not look much like a plane when we zoom in on it. To try to avoid this phenomenon we might decide to work with injective patches. But this leads to a new difficulty. The unit sphere S^2 defined by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

should be a first class example of a surface. But it isn't clear how to write S^2 as the image of an injective regular patch. In fact, using ideas from topology it can be shown that there are no regular injective patches that have S^2 as their trace. The solution to this problem is to use several injective regular patches to describe the surface.

Definition 1.2 (Preliminary) A surface is a set $S \subset \mathbb{R}^3$ such that for every $p \in S$ there is an injective regular patch $\sigma : U \rightarrow \mathbb{R}^3$ satisfying $p \in \sigma(U) \subset S$.

The definition includes such objects as planes and spheres. But it also includes many objects that we would not want to call surfaces. Here are two examples.

Example A. The set $S_A = \mathbb{R}^3$ satisfies the preliminary definition of being a surface. For example, define $\sigma_h : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by $\sigma_h(u, v) = (u, v, h)$. These are regular injective patches and $S_A = \bigcup_{h \in \mathbb{R}} \sigma_h(\mathbb{R}^2)$.

Example B. The figure-of-eight cylinder

$$S_B = \{(\sin(u), \sin(2u), v) : u, v \in \mathbb{R}\}$$

also satisfies the preliminary definition of a surface. To see this, we define $\sigma : (-\pi, \pi) \times \mathbb{R} \rightarrow \mathbb{R}^3$ by

$$\sigma(u, v) = (\sin(u), \sin(2u), v).$$

Then σ is an injective regular patch and the trace of σ is S_B .

Exercise 1.2 Show that the patch σ of Example B is regular and injective, and that its trace is S_B .

Our final definition of a surface will have to be more restrictive than what we have proposed so far. The extra ingredient we need is a little subtle and requires a short diversion into the theory of metric spaces.

1.1 Metric spaces

A metric space is a set S with a distance function $d : S \times S \rightarrow \mathbb{R}$ that satisfies the following properties:

1. For every $p, q \in S$, $d(p, q) \geq 0$ and $d(p, q) = 0$ if and only if $p = q$.
2. For every $p, q \in S$, $d(p, q) = d(q, p)$.
3. For every $p, q, t \in S$, $d(p, t) \leq d(p, q) + d(q, t)$. This is known as the triangle inequality.

The set \mathbb{R}^n with the Euclidean distance function $d(p, q) = \sqrt{\langle p - q, p - q \rangle}$ is a premiere example of a metric space. The only other metric spaces we will work with are subsets of \mathbb{R}^n , with $n = 1, 2$, or 3 . If $S \subset \mathbb{R}^n$, it becomes a metric space simply by inheriting the Euclidean distance function of \mathbb{R}^n .

Let S be a metric space with distance function d . We define the **ball** of radius r centered at $p \in S$ to be the set $B_{S,r}(p) = \{q \in S : d(p, q) < r\}$. When the metric space is clear we will often write $B_r(p)$ instead of $B_{S,r}(p)$. A set $U \subset S$ is **open** if for every $q \in U$, there is a radius r such that $B_r(q) \subset U$. These ideas correspond with the standard notions of balls and open sets when S is \mathbb{R}^n .

Exercise 1.3 Show that balls are open sets. *Hint: Use the triangle inequality.*

Exercise 1.4 Show that if $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets, then the union $U = \bigcup_{\alpha \in I} U_\alpha$ is an open set.

When S is a subset of \mathbb{R}^3 , there is an easy way to characterize the open sets of S .

Lemma 1.1. *Let $S \subset \mathbb{R}^3$ be a metric space with the Euclidean distance, and let U a subset of S . The following conditions are equivalent*

1. U is open.
2. There is an open subset V of \mathbb{R}^3 such that $V \cap S = U$.

Proof: Suppose $U \subset S$ is open. For each $p \in U$ there is a radius r such that $B_{S,r}(p) \subset U$. Since $B_{S,r_p}(p) = \{q \in S : d(p, q) < r_p\}$, and since $B_{\mathbb{R}^3, r_p}(p) = \{q \in \mathbb{R}^3 : d(p, q) < r_p\}$, it follows that $B_{S, r_p}(p) = B_{\mathbb{R}^3, r_p} \cap S$. Now

$$U = \bigcup_{p \in S} B_{S, r_p}(p) = \bigcup_{p \in S} (B_{\mathbb{R}^3, r_p} \cap S) = \left(\bigcup_{p \in S} B_{\mathbb{R}^3, r_p} \right) \cap S.$$

Setting $V = \bigcup_{p \in S} B_{\mathbb{R}^3, r_p}(p)$, we see that V is open and $U = V \cap S$.

On the other hand, if V is an open set in \mathbb{R}^3 and if $p \in S \cap V$, there is a radius r such that $B_{\mathbb{R}^3, r}(p)$ is contained in V . But then $V \cap S \supset (B_{\mathbb{R}^3, r}(p) \cap S) = B_{S, r}(p)$. So $V \cap S$ is an open set in S . \square

If S is a metric space, we say a sequence $\{p_i\}_{i=1}^{\infty}$ **converges** to p (and write $p_i \rightarrow p$) if for every $\epsilon > 0$ there is an integer $N > 0$ such that if $i > N$ then $p_i \in B_{\epsilon}(p)$. Again, when $S = \mathbb{R}^n$, this definition of convergence agrees with the standard definition.

Finally, if $f : X \rightarrow Y$ is a function between two metric spaces, we say that f is **continuous** if whenever a sequence $\{p_i\}_{i=1}^{\infty}$ in X converges to a point p , then the sequence $\{f(p_i)\}_{i=1}^{\infty}$ converges in Y to $f(p)$. This notion of continuity is the standard notion of continuity when the metric space is \mathbb{R}^n .

Exercise 1.5 *Suppose X and Y are metric spaces and $f : X \rightarrow Y$. Suppose $\{U_{\alpha}\}_{\alpha \in I}$ is a collection of open sets such that $\bigcap_{\alpha \in I} U_{\alpha} = X$ and such that the restriction of f to each U_{α} is continuous. Show that $f : X \rightarrow Y$ is continuous.*

An important property of continuous functions is that if $f : X \rightarrow Y$ is continuous and if $U \subset Y$ is open, then $f^{-1}(U)$ is an open set. We prove this fact here for future use.

Proposition 1.2. *If $f : X \rightarrow Y$ is continuous, and if $U \subset Y$ is open, then $f^{-1}(U)$ is open.*

Proof: We proceed by contradiction. Suppose U is an open subset of Y and suppose $f^{-1}(U)$ is not an open subset of X . Then there is a point $p \in f^{-1}(U)$ such that $B_r(p) \not\subset f^{-1}(U)$ for any radius r . In other words, for every radius r there is a point $q \in B_r(p)$ such that $f(q) \notin U$. We construct a sequence by choosing for each $i = 1, 2, \dots$ a point q_i in $B_{\frac{1}{i}}(p)$ such that $f(q_i) \notin U$. It is easily verified that $q_i \rightarrow p$.

Since U is open and since $f(p) \in U$, there is radius r such that $B_r(f(p)) \subset U$. Since f is continuous and since $q_i \rightarrow p$, it follows that $f(q_i) \rightarrow f(p)$. In particular, there is an N such that if $i > N$, then $f(q_i) \in B_r(f(p))$. But then $f(q_i) \in U$ for $i > N$, a contradiction. \square

1.2 The definition of a surface

We are now in a position to properly define a surface.

Definition 1.3 Suppose $S \subset \mathbb{R}^3$. A **chart** into S is a patch $\sigma : U \rightarrow \mathbb{R}^3$ such that

1. σ is regular.
2. σ is injective.
3. The trace of σ is contained in S and there is an open subset V of \mathbb{R}^3 such that $\sigma(U) = V \cap S$.
4. The inverse function $\sigma^{-1} : \sigma(U) \rightarrow U$ is continuous.

The existence of the inverse function supposed in condition 4 follows from the fact that σ is a bijective map from U to $\sigma(U)$. The continuity of the inverse function is the important distinction; verifying this continuity is typically the hardest step in showing a patch is a chart.

Definition 1.2 (Final) A surface S is a non-empty subset of \mathbb{R}^3 such that for every $p \in S$ there is a chart σ into S such that p belongs to the trace of σ .

If we dropped conditions 3 and 4 from the definition of a chart, we would have exactly the preliminary definition of a surface. Condition 3 rules out bad examples such as surface $S_A = \mathbb{R}^3$, and condition 4 rules out the figure-of-eight surface S_B .

We described the set S_A before with the regular injective patches $\sigma_h : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by $\sigma_h(u, v) = (u, v, h)$. These patches fail condition 3 and therefore cannot be charts into S_A . For example, the trace of chart σ_0 is the plane $z = 0$. This set contains the origin, but every open set in \mathbb{R}^3 containing the origin also contains points with $z \neq 0$. This shows that our previous attempt to show S_A is a surface fails the new definition, but we have not proved that S_A is not a surface (there might be some other way to describe S_A with charts). In fact, S_A is not a surface, but a rigorous proof of this will have to wait until later in the course.

We described the surface S_B using a single patch $\sigma : U \rightarrow \mathbb{R}^3$ where $U = (-\pi, \pi) \times \mathbb{R}$ and where

$$\sigma(u, v) = (\sin(u), \sin(2u), v).$$

This patch fails condition 4. Consider the points

$$p_n = \sigma\left(\pi - \frac{1}{n}, 0\right) = \left(\sin\left(\pi - \frac{1}{n}\right), \sin\left(2\pi - \frac{2}{n}\right), 0\right).$$

This is a sequence of points in S_B , and it is easy to see that $p_n \rightarrow (0, 0, 0)$. But $\sigma^{-1}(p_n) = (\pi - \frac{1}{n}, 0)$. The sequence $\{\sigma^{-1}(p_n)\}_{n=1}^{\infty}$ does not converge to any point of U (it converges in \mathbb{R}^2 to $(\pi, 0)$ which is outside the domain U). Again, we have not shown here definitively that S_B is not a surface; the proof that this is true will also have to wait.

1.3 Showing a set is a surface

Showing a set is a surface from the definition can be cumbersome, especially the verification that the inverse function of a chart is continuous. So it is useful to have some alternate means of description.

We first show that graphs of functions are surfaces. Suppose U is an open subset of \mathbb{R}^2 and $f : U \rightarrow \mathbb{R}$ is a smooth function. Let $S = \{(x, y, z) : (x, y) \in U, z = f(x, y)\}$. To see that S is a surface, we define $\sigma : U \rightarrow \mathbb{R}^3$ by $\sigma(u, v) = (u, v, f(u, v))$. The Jacobian matrix of σ is

$$D\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_u & f_v \end{pmatrix}$$

which has rank 2. So σ is regular and satisfies condition 1 of being a chart into S . It is easy to see that σ is injective and therefore satisfies condition 2. The trace of σ is contained in S and is in fact all of S . So condition 3 is satisfied with $V = \mathbb{R}^3$. Finally, we need to show that σ^{-1} is continuous. To do this, we define $p : U \times \mathbb{R} \rightarrow U$ by $p(x, y, z) = (x, y)$. The following exercise shows that p is continuous.

Exercise 1.6 Suppose U and V are open sets in \mathbb{R}^2 and \mathbb{R}^3 respectively and that $p : V \rightarrow U$ is a continuous function. Show that if $S \subset V$ is a set, then the restriction of p to S , $p|_S : S \rightarrow U$, is also continuous.

Since $p|_S$ is exactly the function σ^{-1} , we conclude that σ^{-1} is continuous. So σ is a chart into S . Since the trace of σ is all of S , we have shown S is a surface.

Of course, some surfaces cannot be described by a single chart. The following propositions shows that we can show a set is a surface by breaking it up into certain pieces and showing that each piece can be written as the graph of a function.

Proposition 1.3. Suppose $f : U \rightarrow \mathbb{R}$ is a smooth function, where U is an open subset of \mathbb{R}^2 . Suppose also that $\sigma : U \rightarrow \mathbb{R}^3$ is given by one of the following three expressions:

$$\sigma(u, v) = (u, v, f(u, v)) \tag{1}$$

or

$$\sigma(u, v) = (u, f(u, v), v) \tag{2}$$

or

$$\sigma(u, v) = (f(u, v), u, v). \tag{3}$$

If $S \subset \mathbb{R}^3$ is a set, and if there exists an open set V in \mathbb{R}^3 such that $\sigma(U) = V \cap S$, then σ is a chart into S .

Proof: We will assume that $\sigma(u, v) = (u, v, f(u, v))$, as the proof in the other cases proceeds in the same way interchanging the roles of the variables. We can also assume that V satisfies $V \subset U \times \mathbb{R}$. Otherwise we can replace V with $V \cap (U \times \mathbb{R})$. This is still an open subset of \mathbb{R}^3 and $V \cap (U \times \mathbb{R}) \cap S = \sigma(U) \cap (U \times \mathbb{R}) = \sigma(U)$.

Arguing as before, σ satisfies conditions 1 and 2 of being a chart into S , and it satisfies condition 3 by hypothesis. So we just need to check that σ^{-1} is continuous. Let $p : V \rightarrow U$ be the projection function $p(x, y, z) = (x, y)$ used before. Notice that the image of p is contained in U since $V \subset U \times \mathbb{R}$. From Exercise 1.6 we conclude that $p|_{\sigma(U)}$ is continuous, and this is exactly σ^{-1} . Hence σ is a chart into S . \square

Exercise 1.7 Let $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Define patches $\sigma_i : U \rightarrow \mathbb{R}^3$, $i = 1, \dots, 6$, as follows:

$$\begin{aligned}\sigma_1(u, v) &= (u, v, \sqrt{1 - u^2 - v^2}) \\ \sigma_2(u, v) &= (u, v, -\sqrt{1 - u^2 - v^2}) \\ \sigma_3(u, v) &= (u, \sqrt{1 - u^2 - v^2}, v) \\ \sigma_4(u, v) &= (u, -\sqrt{1 - u^2 - v^2}, v) \\ \sigma_5(u, v) &= (\sqrt{1 - u^2 - v^2}, u, v) \\ \sigma_6(u, v) &= (-\sqrt{1 - u^2 - v^2}, u, v)\end{aligned}$$

Use Proposition 1.3 and these patches to show that the sphere S^2 is a surface.

Describing surfaces by piecing together graphs of functions can still be tedious. We now turn to a way of describing surfaces that often works with minimal effort.

If V is an open set in \mathbb{R}^3 , and if $g : V \rightarrow \mathbb{R}$, we say that $p \in V$ is a **regular point** of g if $Dg_p \neq 0$. Otherwise we say p is a **critical point** (of g). Finally, we say that $c \in \mathbb{R}$ is a **regular value** of g if $g^{-1}(c) \neq \emptyset$ and if each $p \in g^{-1}(c)$ is a regular point of g (otherwise c is a **critical value**).

Proposition 1.4. Suppose V is an open subset of \mathbb{R}^3 and $g : V \rightarrow \mathbb{R}$ is a smooth function. If c is a regular value of g , then $g^{-1}(c)$ is a surface. Moreover, for every point $p \in g^{-1}(c)$ there is a chart σ into S such that p is in the trace of σ and σ is of the form (1), (2), or (3).

Proof: Since $Df_p \neq 0$, we can assume that $\frac{\partial f}{\partial z}(p) \neq 0$ (otherwise we could just rename the coordinates). We define a function $F : V \times \mathbb{R} \rightarrow \mathbb{R}^3$ by $F(x, y, z) = (x, y, g(x, y, z))$.

Then

$$DF = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g_x & g_y & g_z \end{pmatrix}.$$

Since $g_z(p) \neq 0$, DF_p is an invertible matrix. The inverse function theorem implies that there exists an open set $W \subset V$ containing p such that F is a bijection from W to $F(W)$ and such that the inverse function of F defined on W is smooth. Notice that the inverse function must have the form $F^{-1}(x, y, z) = (x, y, f(x, y, z))$ for some smooth function f .

Let $U = \{(u, v) \in \mathbb{R}^2 : (u, v, c) \in F(W)\}$ where $c = f(p)$. Since $F(W)$ is open in \mathbb{R}^3 , this easily seen to be an open set in \mathbb{R}^2 . We define a smooth patch $\sigma : U \rightarrow \mathbb{R}^3$ by $\sigma(u, v) = (u, v, f(u, v, c))$. Then $\sigma(U) = g^{-1}(c) \cap W$. From Proposition 1.3 we conclude that σ is a chart into $g^{-1}(c)$ with p in its trace. Since the choice of p is arbitrary, we conclude $g^{-1}(c)$ is a surface. \square

We can now give an easy proof that the sphere is a surface. Let $g(x, y, z) = x^2 + y^2 + z^2$. The sphere is the set $g^{-1}(1)$. Now $Dg = (2x, 2y, 2z)^T$. In particular, Dg vanishes only at $(0, 0, 0)$ which does not belong to $g^{-1}(1)$. Hence the sphere is a surface.

The key technical tool we used in proving Proposition 1.4 is the Inverse Function Theorem from multidimensional calculus. Intuitively, this theorem states that if $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is smooth, and if its linear approximation at some point $p \in U$ is an invertible function, then there is a neighborhood U_p of p such that F restricted to U_p is itself an invertible function (with smooth inverse). Of course, we have to be careful about the domains and ranges these functions, and the formal result is the following.

Proposition 1.5. Inverse Function Theorem *Let U be an open set of \mathbb{R}^n and suppose $F : U \rightarrow \mathbb{R}^n$ is a smooth function. If for some point $p \in U$ the Jacobian matrix DF_p is invertible, then there is an open set $U_p \subset U$ containing p such that*

1. $F(U_p) \subset \mathbb{R}^n$ is an open set.
2. The restriction of F to U_p is bijective and has a smooth inverse function. That is, $F : U_p \rightarrow F(U_p)$ has a smooth inverse function $F^{-1} : F(U_p) \rightarrow U_p$.

2. Smooth functions on surfaces

Now if A is an arbitrary subset of \mathbb{R}^n , we do not have a good notion of what it means for a function $f : A \rightarrow \mathbb{R}$ to be smooth. The definition of the derivative at a point p requires f be defined on an open set containing p . So we have until now restricted our discussion of smooth functions to function defined on open sets. To define smooth functions on surfaces, we need a new idea.

Definition 2.3 Let S be a surface. We say that $f : S \rightarrow \mathbb{R}^n$ is **smooth** if for every chart $\sigma : U \rightarrow S$, $f \circ \sigma : U \rightarrow \mathbb{R}^n$ is a smooth function.

The motivation for this definition comes from the intuition that a chart should be a smooth function into S and that the composition of smooth maps should be smooth. But if S is a surface, what does it mean for a function taking values in S to be smooth? One possible definition would be $f : U \subset \mathbb{R}^n \rightarrow S$ is smooth if it is smooth thought of as a function taking values in \mathbb{R}^3 . This would be an acceptable definition, for now, but it turns out that it doesn't generalize well. Instead, we follow the approach of Definition 2.3.

Definition 2.4 Let S be a surface, and let $U \subset \mathbb{R}^n$ be an open set. We say that $f : U \rightarrow S$ is **smooth** if for every chart $\sigma : U \rightarrow S$, $\sigma^{-1} \circ f$ is smooth on its domain $f^{-1}(\sigma(U))$.

What is the domain of $f^{-1}(\sigma(U))$? Since $\sigma(U)$ is the intersection of an open set V with S , it follows that $f^{-1}(\sigma(U)) = f^{-1}(V)$. Since f is continuous, $f^{-1}(V)$ is an open subset of U . It might be empty, in which case we say $\sigma^{-1} \circ f$ is trivially smooth. Otherwise, it makes sense to talk about a smooth function from $f^{-1}(\sigma(U))$ to U . So this definition makes sense.

Exercise 2.8 Show that if $U \subset \mathbb{R}^n$ is open and if $f : U \rightarrow S$ is smooth, then f is smooth thought of as a function taking values in \mathbb{R}^3 .

The converse of Exercise 2.8 is also true. The following consequence of the Inverse Function Theorem is the key ingredient to show this.

Lemma 2.1. $\sigma : U \rightarrow \mathbb{R}^3$ is a regular patch and suppose $q_0 \in U$. Then there is an open set $U_{q_0} \subset U$ containing q_0 , an open set $V \subset \mathbb{R}^3$ containing $f(q_0)$, and a smooth function $G : V \rightarrow U_{q_0}$ such that $G(\sigma(q)) = q$ for every $q \in U_{q_0}$.

Proof: Let $q_0 = (u_0, v_0)$. Since $\sigma_u(u_0, v_0)$ and $\sigma_v(u_0, v_0)$ are linearly independent, we can assume (possibly after relabeling coordinates) that these vectors together with $e_3 = (0, 0, 1)^T$ form a basis for \mathbb{R}^3 . Define $F : U \times \mathbb{R} \rightarrow \mathbb{R}^3$ by $F(u, v, w) = \sigma(u, v) + (0, 0, w)$. The Jacobian matrix of F is $DF = [\sigma_u | \sigma_v | e_3]$. In particular, $DF_{(u_0, v_0, 0)}$ is an invertible matrix. The Inverse Function Theorem implies that there is an open set W containing $(u_0, v_0, 0)$ such that F is a bijective map from W to an open set $V = F(W)$, and such that the inverse function from V to W is smooth. By shrinking W we can assume it is of the form $U_{q_0} \times (-\epsilon, \epsilon)$ where U_{q_0} is an open subset of U containing q_0 .

Define $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by $\pi(x, y, z) = (x, y)$, and let $G = \pi \circ F^{-1}$. Then G is a smooth

function taking V to U_{q_0} , and if $(u, v) \in U_{q_0}$, then

$$\begin{aligned} G(\sigma(u, v)) &= G(F(u, v, 0)) \\ &= \pi(F^1(F(u, v, 0))) \\ &= \pi(u, v, 0) \\ &= (u, v). \end{aligned}$$

So G is the function we seek. \square

Proposition 2.2. *Let S be a surface. Suppose $V \subset \mathbb{R}^n$ is open and suppose $f : V \rightarrow \mathbb{R}^3$ is smooth. If $F(V) \subset S$, then $f : V \rightarrow S$ is smooth.*

Proof: We need to verify that if $\sigma : U \rightarrow S$ is a chart, then $\sigma^{-1} \circ f$ is smooth on its domain. Let σ be a chart and suppose x is a point in the domain of $\sigma^{-1} \circ f$. We need to show that $\sigma^{-1} \circ f$ is smooth on a neighborhood of x . Let $p = f(x)$ and let $q = \sigma^{-1}(p)$. By Lemma 2.1 there is an open subset U_q of U containing q and an open subset V_p of \mathbb{R}^3 containing p and a function $G : V_p \rightarrow U_q$ such that $G \circ \sigma : U_q \rightarrow U_q$ is the identity. In particular, G restricted to $V_p \cap S$ is σ^{-1} . But then on $f^{-1}(V_p)$ (which is an open set containing x) we have $\sigma^{-1} \circ f = G \circ f$ which is smooth. \square

Suppose $\sigma : U \rightarrow S$ and $\hat{\sigma} : \hat{U} \rightarrow S$ are charts into S such that $\sigma(U) \cap \hat{\sigma}(\hat{U}) \neq \emptyset$. By shrinking domains we can assume that $\sigma(U) = \hat{\sigma}(\hat{U})$. We call the function $\hat{\sigma}^{-1} \circ \sigma : U \rightarrow \hat{U}$ a **transition function**. Proposition 2.2 implies transition functions are smooth.

Proposition 2.3. *Let S be a surface. Suppose $\sigma : U \rightarrow S$ and $\hat{\sigma} : \hat{U} \rightarrow S$ are two charts such that $\sigma(U) = \hat{\sigma}(\hat{U})$. Then $\hat{\sigma}^{-1} \circ \sigma : U \rightarrow \hat{U}$ is a smooth function with a smooth inverse function.*

Proof: Since $\sigma : U \rightarrow S$ is smooth as a function into \mathbb{R}^3 , Proposition 2.2 implies $\hat{\sigma}^{-1} \circ \sigma$ is smooth on its domain (which is all of U). So $\hat{\sigma}^{-1} \circ \sigma$ is smooth. This same argument implies $\sigma^{-1} \circ \hat{\sigma} : \hat{U} \rightarrow U$ is also smooth, and it is easy to see this is the inverse function of $\hat{\sigma}^{-1} \circ \sigma$. \square

Definitions 2.3 and 2.4 are aesthetically pleasing, but on the face of things they would be hard to work with in practice. To show a function $f : S \rightarrow \mathbb{R}$ is smooth, it would seem that we would have to check that the composition of f with every possible chart is smooth. Fortunately, it turns out that we do not need to check that $f \circ \sigma$ is smooth for every chart, but only for a select few.

Definition 2.5 *Let S be a surface. An **atlas** on S is a collection of charts $\{\sigma_\alpha\}_{\alpha \in I}$ on S with domains U_α such that $\cup_{\alpha \in I} \sigma_\alpha(U_\alpha) = S$.*

An example of an atlas on the sphere is the collection of six charts from Exercise 1.7. The

following proposition shows that a function on S is smooth if its composition with each chart in an atlas is smooth.

Proposition 2.4. *Let S be a surface, and let $\{\sigma_\alpha\}_{\alpha \in I}$ be an atlas for S . If $f : S \rightarrow \mathbb{R}^n$, then f is smooth if and only if $f \circ \sigma_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ is smooth for each α .*

Proof: If f is smooth, then the composition of f with any chart is smooth. On the other hand, suppose $f \circ \sigma_\alpha$ is smooth for each α . Let $\sigma : U \rightarrow S$ be some chart into S . Let $q \in U$, and let α be an index such that $\sigma(q)$ is in the trace of σ_α . We need to show that $f \circ \sigma$ is smooth on an open set containing q . Shrinking the domain U to be a small open set containing q if necessary, we can suppose that $\sigma(U) \subset \sigma_\alpha(U_\alpha)$. Now $f \circ \sigma = f \circ \sigma_\alpha \circ \sigma_\alpha^{-1} \circ \sigma$. Since $f \circ \sigma_\alpha$ is smooth by hypothesis, and since $\sigma_\alpha^{-1} \circ \sigma$ is smooth (it is a transition function) we conclude that $f \circ \sigma$ is smooth also. \square

We also have the following related result. The proof is left to the reader.

Proposition 2.5. *Let S be a surface, and let $\{\sigma_\alpha : U_\alpha \rightarrow S\}_{\alpha \in I}$ be an atlas for S . If U is an open subset of \mathbb{R}^n and if $f : U \rightarrow S$, then f is smooth if and only if $\sigma_\alpha^{-1} \circ f$ is smooth on its domain for each α .*

Exercise 2.9 *Suppose $V \subset \mathbb{R}^3$ is an open set containing a surface S , and suppose $f : V \rightarrow \mathbb{R}^n$ is a continuous function. Show that the restriction of f to S is also continuous.*

What about functions between surfaces? Following the model we have already established, the following definition is very natural.

Definition 2.6 *Let S and \hat{S} be two surfaces. We say a function $f : S \rightarrow \hat{S}$ is smooth if it is continuous and if $\hat{\sigma}^{-1} \circ f \circ \sigma$ is smooth on its domain for every pair of charts $\sigma : U \rightarrow S$ and $\hat{\sigma} : \hat{U} \rightarrow \hat{S}$.*

An easy recipe for constructing a smooth map from S to \hat{S} is as follows. Start with a smooth function f from an open set V containing S such that its image is contained in \hat{S} . Then the restriction of f to S thought of as a map into \hat{S} is smooth.

Definition 2.7 *Let S and \hat{S} be two surfaces. We say a function $f : S \rightarrow \hat{S}$ is **diffeomorphism** smooth if it is smooth and has a smooth inverse function*

We end this section with the following consequence of Lemma 2.1. Proposition 1.4 gives a powerful way for constructing surfaces. But it does not directly construct charts into the surfaces it constructs. But charts are valuable to have for doing computations. Typically the hardest part of showing a regular patch σ taking values in S is a chart is showing that its inverse function is smooth. If we know in advance that S is a surface however, it turns out we can skip this step.

Proposition 2.6. *Suppose S is a surface and suppose $\sigma : U \rightarrow \mathbb{R}^3$ is a patch such that*

1. σ is regular,
2. σ is injective,
3. $\sigma(U) \subset S$.

Then σ is a chart into S .

Proof: By hypothesis, σ satisfies conditions 1 and 2 of being a chart, so we only need to verify conditions 3 and 4. Consider a point $p \in U$. From Lemma 2.1 there is an open set U_p containing p , an open subset $V_p \subset \mathbb{R}^3$ containing $\sigma(p)$, and a smooth function $G_p : V_p \rightarrow U_p$ such that $G_p(\sigma(q)) = q$ for every $q \in U_p$.

Let $\hat{\sigma} : \hat{U} \rightarrow S$ be a chart into S such that $\sigma(p) \in \hat{\sigma}(\hat{U})$. Since $\hat{\sigma}$ is a chart, $\hat{\sigma}(\hat{U}) = \hat{V}_p \cap S$ for some open set $\hat{V}_p \subset \mathbb{R}^3$. Let $W_p = V_p \cap \hat{V}_p$. By shrinking domains if need be we can assume that $\hat{U} = \hat{\sigma}^{-1}(W_p)$ and $U_p = \sigma^{-1}(W_p)$. Since $\hat{\sigma}$ is a chart, it follows that $\hat{\sigma}(\hat{U}) = S \cap W_p$. Hence $W_p \cap S = \sigma(U_p)$. Taking the union over all points p , we find that $\sigma(U)$ is contained in $\cup_{p \in U} W_p$ which is an open set. So condition 3 of being a chart is satisfied.

Finally, on $W_p \cap S$, σ^{-1} is the same as G_p . Since G_p is continuous, so is its restriction to $W_p \cap S$. So σ^{-1} is continuous on each open subset $W_p \cap S$ of $\sigma(U)$. Exercise 1.5 then shows that σ^{-1} is continuous on all of $\sigma(U)$. \square