Bounds on the metric and partition dimensions of a graph

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October 20, 2005

2000 Mathematics Subject Classification: 05C12, 05C15, 05C62

Abstract

Given a graph G, we say $S \subseteq V(G)$ is resolving if for each pair of distinct $u, v \in V(G)$ there is a vertex x in S where $d(u, x) \neq d(v, x)$. The metric dimension of G is the minimum cardinality of all resolving sets. For $w \in V(G)$, the distance from w to S, denoted $d(w, S)$, is the minimum distance between w and the vertices of S. Given $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ an ordered partition of $V(G)$ we say $\mathcal{P}$ is resolving if for each pair of distinct $u, v \in V(G)$ there is a part P_i where $d(u, P_i) \neq d(v, P_i)$. The partition dimension is the minimum order of all resolving partitions. In this paper we consider relationships between metric dimension, partition dimension, diameter, and other graph parameters. We construct "universal examples" of graphs with given partition dimension, and we use these to provide bounds on various graph parameters based on metric and partition dimensions. We form a construction showing that for all integers α and β with $3 \leq \alpha \leq \beta + 1$ there exists a graph G with partition dimension α and metric dimension β, answering a question of Chartrand, Salehi, and Zhang [3].

1 Introduction

Graphs will be simple, undirected, and finite with at least two vertices. For undefined terms and concepts the reader is referred to [1]. Given G with a vertex v and an ordered set of vertices $S = \{v_1, v_2, \ldots, v_k\}$ the *metric representation* of v with respect to S is the vector $\mathbf{r}(v|S) := (d(v, v_1), d(v, v_2), \ldots, d(v, v_k))$. We say S *differentiates* vertices u and v if $\mathbf{r}(u|S) \neq \mathbf{r}(v|S)$. A set is *resolving* if it differentiates each pair of distinct vertices. Elements in a resolving set will be called *landmarks*. The *metric dimension* of G, denoted $\dim_{\mathrm{M}}(G)$, is the minimum order of all resolving sets. Metric dimension is studied in a variety of places including [4, 6, 8, 9] where it is sometimes referred to as the "location number".

By "partition" we will mean an ordered partition. Given $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$, a partition of $V(G)$, the *partition representation* of v with respect to $\mathcal{P}$ is the vector $\mathbf{r}(v|\mathcal{P}) = (d(v, P_1), d(v, P_2), \ldots, d(v, P_k))$. We say $\mathcal{P}$ *differentiates* vertices u and v if $\mathbf{r}(v|\mathcal{P}) \neq \mathbf{r}(u|\mathcal{P})$. We say $\mathcal{P}$ is *resolving* if it differentiates each pair of distinct vertices. The *partition dimension*, $\dim_{\mathrm{P}}(G)$, is the minimum order of all resolving partitions. Partition dimension seems to have been introduced in [2] and further studied in [3].

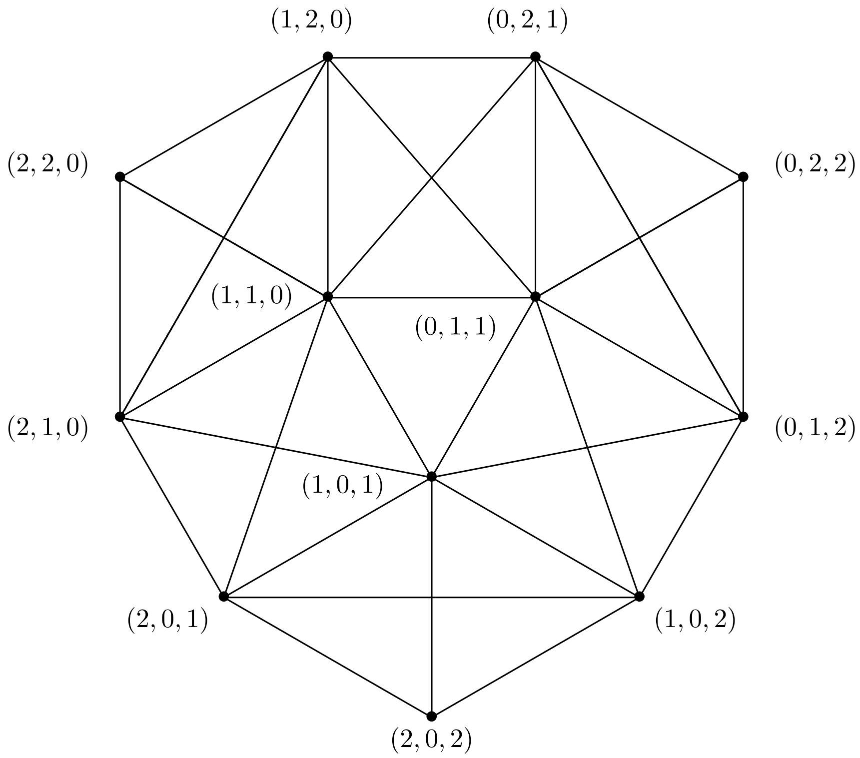

Figure 1: The graph $H(3, 2)$. Vertices are labeled with the corresponding vectors.

By avoiding disconnected graphs we ensure that all distances are finite. Thus, when speaking of metric and partition dimensions of graphs, we will restrict ourselves to connected graphs.

In this paper, we consider bounds on metric and partition dimension in terms of each other and other graphical parameters.

2 A Universal Example

We now construct a graph $H(p, s)$ with partition dimension p, having the property that each graph with partition dimension p and diameter at most s is a subgraph of $H(p, s)$.

Example 2.1. Given positive integers p, s, with $p \geq 2$, let $H(p, s)$ be the graph whose vertex set is those vectors in $\{0, 1, 2, \ldots, s\}^p$ that contain exactly one zero. Let two vertices be adjacent if they differ by at most one in each coordinate.

The graph $H(3, 2)$ is represented in Figure 1; each vertex is labeled with the corresponding vector. Figure 2 shows $H(3, 6)$.

Now, suppose G is a graph with diameter at most s. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$ be a resolving partition of $V(G)$. Let us say the *natural embedding* of G into $H(p, s)$ with respect to $\mathcal{P}$ is the mapping that takes each vertex v to $\mathbf{r}(v|\mathcal{P})$. We note that if G has a natural embedding into $H(p, s)$, then $\dim_{\mathrm{P}}(G) \leq p$. The mapping $r\colon V(G) \to V(H(p, s))$ is injective and preserves adjacencies, although it may not preserve nonadjacencies.

Now, given $H(p, s)$, let P_i be those vertices with a zero in their ith coordinate. Suppose G is a subgraph of $H(p, s)$. Let Q_i be $V(G) \cap P_i$. Let us say G is a *natural subgraph* of $H(p, s)$ if for each vertex v and for each Q_i where $v \notin Q_i$, there exists a vertex u adjacent in G to v that is closer in $H(p, s)$ to Q_i.

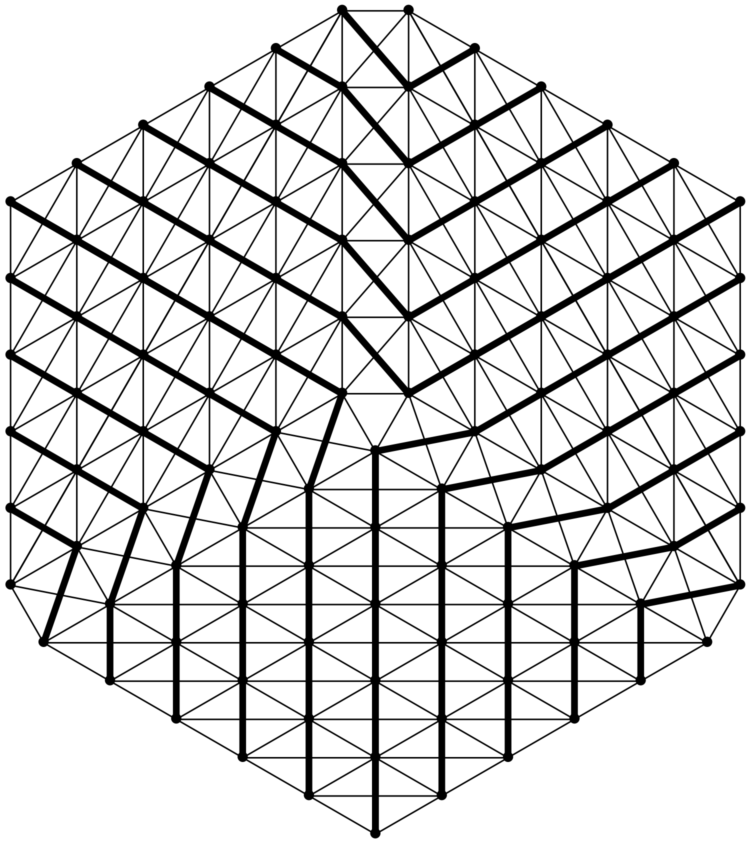

Figure 2: The graph $H(3, 6)$, shown with a partition of its edge set into a planar graph and a linear forest.

Note that, if G is a natural subgraph, then each Q_i must be nonempty. Further, $\{Q_1, Q_2, \dots, Q_p\}$ is a resolving partition of $V(G)$, and each vertex of G (where a vertex is considered as a vector) is equal to its own partition representation. It follows that a subgraph G of $H(p,s)$ is a natural subgraph if and only if the mapping that takes v to v is a natural embedding.

Observation 2.2. *Let G be a graph. If $\dim_{\mathrm{P}}(G) = p$, and s is at least the diameter of G, then G is isomorphic to a natural subgraph of $H(p,s)$.*

Observation 2.3. *If a graph G is isomorphic to a natural subgraph of $H(p,s)$, then $\dim_{\mathrm{P}}(G) \leq p$.*

Observation 2.4. *For all positive integers p, s, with $p \geq 2$, we have $\dim_{\mathrm{P}}(H(p,s)) = p$.*

We use this "universal example" in some of the subsequent proofs. In each remark, this family of graphs can be used to form an intuitive picture that illustrates the dynamics of the result.

3 Some Simple Bounds

Theorem 3.1. *If a graph G has maximum degree Δ and partition dimension p then*

$$\Delta \leq 3^{p-1} - 1.$$

Proof. By Observation 2.2 we need only show the bound holds for all $H(p,s)$. Consider a vertex v in $H(p,s)$. Some coordinate of v is zero, say the first. Suppose u is adjacent to v. Consider any coordinate of u other than the first. This differs from the corresponding coordinate of v by at most one, so it has one of three possible values. If coordinates 2 through p of u are nonzero, then the first coordinate of u is zero. Otherwise, the first coordinate must be one. Thus, the first coordinate of u is determined by the remaining $p-1$ coordinates. Since not all coordinates in u and v are the same, there are at most $3^{p-1} - 1$ vertices adjacent to v. The desired bound now follows. $\square$

The graphs $H(p,s)$ show that the bound in Theorem 3.1 is sharp. We may also restate this bound: For G an arbitrary graph, $\dim_{\mathrm{P}}(G) \geq 1 + \log_3(\Delta + 1)$.

Suppose S is a set of landmarks. We can form a resolving partition by putting each vertex of S in a singleton and placing all other vertices in a separate part. Hence, as is observed in several places, $\dim_{\mathrm{P}}(G) \leq 1 + \dim_{\mathrm{M}}(G)$. Applying the previous theorem we obtain an alternate proof to the following result of [5].

Corollary 3.2 (Chartrand, Poisson, Zhang 2000 [5]). *If G is a graph with maximum degree Δ, then $\dim_{\mathrm{M}}(G) \geq \log_3(\Delta + 1)$.*

Sharpness for this bound is given in [5].

Theorem 3.3. *If a graph G has clique number ω and partition dimension p then*

$$\omega \leq 2^{p-1}.$$

Proof. By Observation 2.2 we need only show the bound holds for all $H(p,s)$. Let $T \subseteq V(G)$ be a set of pairwise adjacent vertices in $H(p,s)$. We show that $|T| \leq 2^{p-1}$.

For $1 \leq i \leq p$, the ith coordinates of each pair of vertices in T (considered as vectors) must differ by at most one. Hence, at most two values are used in each coordinate of vertices in T.

Let $u \in T$. Some coordinate of u is zero, say the first. As in the proof of Theorem 3.1, it follows that the first coordinate of each vertex in T is determined by the values of the other $p-1$ coordinates. Since there are at most two choices for each of coordinates 2 through p, we have $|T| \leq 2^{p-1}$. $\square$

If $s \geq 2$, and we consider $H(p,s)$, vectors of the form $(0, x_2, x_3, \dots, x_p)$ with $1 \leq x_i \leq 2$, form a clique of order 2^{p-1}. By Observation 2.4 we have $\dim_{\mathrm{P}}(H(p,s)) = p$, and so the bound in Theorem 3.3 is sharp. As with Theorem 3.1, we can restate this bound: $\dim_{\mathrm{P}}(G) \geq 1 + \log_2(\omega)$.

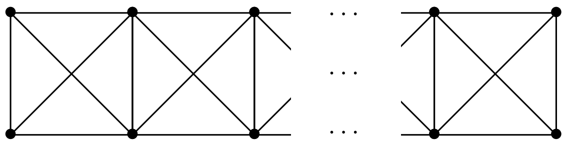

Figure 3: A family of graphs of arbitrarily high diameter, for which $\dim_{\mathrm{M}}(G+H) = \dim_{\mathrm{M}}(G)+\dim_{\mathrm{M}}(H)$.

Given disjoint graphs G and H, the *join* of G and H, denoted $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ together with $\{\, uv : u \in V(G) \wedge v \in V(H) \,\}$. In [6] we find the formula $\dim_{\mathrm{M}}(G + H) = \dim_{\mathrm{M}}(G) + \dim_{\mathrm{M}}(H)$, for all graphs G and H. But if we let G and H be complete graphs, and note that $\dim_{\mathrm{M}}(K_n) = n - 1$, then we see that this remark is less than accurate. However, we can observe the following.

Remark 3.4. For all graphs G and H,

$$\dim_{\mathrm{M}}(G + H) \geq \dim_{\mathrm{M}}(G) + \dim_{\mathrm{M}}(H).$$

Proof. Suppose S is a set of landmarks in $G + H$. The diameter of $G + H$ is at most two. Hence, the distance in $G+H$ between a vertex in $V(G)$ and an element of $S \cap V(G)$ is zero, one, or two depending on whether v is in S, adjacent to an element of $S \cap V(G)$, or otherwise. The distance in G between a vertex $v \in V(G)$ and a vertex $u \in S \cap V(G)$ is zero, one, or at least two depending upon whether $d_{G+H}(v, u)$ is zero, one, or exactly two. The same relation holds between vertices of H and $S \cap V(H)$. Hence, $S \cap V(G)$ is a set of landmarks for G and $S \cap V(H)$ is a set of landmarks for H. $\square$

For most pairs of graphs we have checked, equality does not hold in the above remark. Clearly, if G and H both have diameter two then equality must hold. Suppose G and H are graphs belonging to the family described in Figure 3. Then $\dim_{\mathrm{M}}(G + H) = \dim_{\mathrm{M}}(G) + \dim_{\mathrm{M}}(H)$. Hence, there exist pairs of graphs with arbitrarily large diameter where equality in Remark 3.4 holds.

4 Graphs With Low Dimension

If G has a vertex of degree greater than two, then by Theorem 3.1 it must have partition dimension at least three. Hence, if the partition dimension equals two the graph must be a cycle or path. Further, if G has $\{P_1, P_2\}$ as a resolving partition, there cannot be two or more edges between P_1 and P_2, for otherwise two vertices in one part would be distance one away from the other part. Thus, we have an alternative proof to the following result of [3].

Remark 4.1 (Chartrand, Salehi, Zhang 2000 [3]). A connected graph G is a path if and only if $\dim_{\mathrm{P}}(G) = 2$.

Graphs with partition dimension three do not have such explicit structure. However, we show below that a good deal of structure does exist.

Theorem 4.2. *Let G be a graph. If $\dim_{\mathrm{P}}(G) = 3$ then G has thickness at most two.*

Proof. By Observation 2.2, it suffices to show that $H(3, s)$ has thickness at most two. We reference Figure 2 where the edge set of $H(3, 6)$ is shown decomposed into two planar subgraphs. The dark edges in the figure form a linear forest. The remaining edges form a planar subgraph. Clearly, $H(3, s)$ can be similarly decomposed for all s, which establishes our result. $\square$

Note that $H(3, 6)$, in Figure 2, is nonplanar. Thus, the above result is sharp.

Graphs with partition dimension three have other properties that are easily established by observing that they hold for $H(3, s)$. For example, they have clique number at most four and do not contain $K_{3,3}$

as a subgraph. (Clique size could also be bounded using Theorem 3.3.) Noting that a graph with metric dimension at most two must have partition dimension at most three, we obtain a simple proof of the following result of [7].

Remark 4.3 (Khuller, Raghavachari, Rosenfeld 1996 [7]). If a graph G has metric dimension two then G cannot have K_5 nor $K_{3,3}$ as a subgraph.

5 Coloring

A graph with bounded partition dimension also has bounded maximum degree, by Theorem 3.1. Hence, it has bounded chromatic number. The same is true for graphs with bounded metric dimension. In this section we produce tight bounds.

Theorem 5.1. *If a graph G has metric dimension m, then*

$$\chi(G) \leq 2^m.$$

Proof. Let S be a set of landmarks of order m. Given a vertex v, let $c(v)$ be $\mathbf{r}(v|S)$ (modulo 2). If u and v are adjacent, then $\mathbf{r}(u|S)$ and $\mathbf{r}(v|S)$ differ by exactly one in some coordinate; $c(u)$ and $c(v)$ thus differ in this coordinate. Hence, thinking of vectors as colors, c is a proper coloring of G. As there are m coordinates, each with one of two values, G is 2^m-colorable. □

In order to establish sharpness, we now construct a graph denoted $J(k)$. Let A be those vectors in $\{0,1\}^k$ that contain exactly one zero. Let B be the set $\{1,2\}^k$. The vertex set will be the union of A and B. Let two vertices be adjacent if, when considered as vectors, they differ by at most one in each coordinate. Note that the set B induces a clique of order 2^k. Further, A forms a resolving set.

Remark 5.2. The graph $J(k)$ has metric dimension k and chromatic number 2^k.

Proof. As in the proof of the preceding theorem, we can take the entries of each vertex modulo 2. This creates a coloring with 2^k colors. As this equals the clique size of $J(k)$, we see $\chi(J(k)) = 2^k$. As A forms a resolving set, $\dim_{\mathrm{M}}(J(k)) \leq k$. By Theorem 5.1, if $\dim_{\mathrm{M}}(J(k))$ is less than k, then the chromatic number of $J(k)$ is less than 2^k, a contradiction. □

Thus, Theorem 5.1 is sharp.

In the following, to *flip* a bit means to change its value. Hence, flipping 0 changes it to 1 and vice-versa.

Theorem 5.3. *If a graph G has partition dimension p, then*

$$\chi(G) \leq 2^{p-1}.$$

Proof. Let G be such a graph. We construct a proper coloring of G using at most 2^{p-1} colors.

Let $\mathcal{P}$ be a resolving partition of $V(G)$ where $|\mathcal{P}| = \mathrm{p}$. Label each vertex of G with its partition representation modulo 2. Each label is a binary vector of length p. Now modify each vertex's label as follows. First flip all the bits lying to the left of the position corresponding to the part the vertex lies in. Then, if the first bit in the resulting vector is 1, flip *all* the bits in the vector; otherwise leave all the bits alone.

(As an example, suppose we number the positions from 1 to p. Say the partition representation modulo 2 for some vertex is 0101, and this vertex lies in part 3 of the partition. We flip the bits to the left of position 3 to obtain 1001. Since the first bit is a 1, we now flip all the bits to obtain the final label: 0110.)

The resulting labeling of the vertices is our coloring. Each vertex is labeled with a binary vector of length p, whose first bit is a 0. Thinking of vectors as colors, there are at most 2^{p-1} colors used. It remains to show that this is a proper coloring.

Let x and y be adjacent vertices in G. We show that x and y receive distinct colors. There are two cases to consider: In one case, x and y lie in the same part of $\mathcal{P}$. In the other case, x and y lie in different parts.

Case 1: Suppose x and y lie in the same part of $\mathcal{P}$.

Say x and y lie in part i of $\mathcal{P}$. Since x and y are adjacent, they must have different partition representations modulo 2. When we flip all bits to the left of position i, the resulting vectors are still different. Now, if neither x nor y get all their bits flipped, then they receive different colors. Similarly, if they both get all their bits flipped, then they receive different colors. If one gets all its bits flipped and the other does not, then their colors must have different values in position i, since both began with a 0 in this position. Hence, x and y receive distinct colors.

Case 2: Suppose x and y lie in different parts of $\mathcal{P}$.

Say without loss of generality that x lies in part i and y lies in part j, with $i < j$. Because x and y are adjacent, the values in positions i, j of x's partition representation modulo 2 are 0, 1, respectively. Further, y's values are 1, 0. After flipping the bits to the left of the position corresponding to the part a vertex lies in, x has 0, 1, while y has 0, 0 in the ith and jth coordinates. Thus, regardless of whether each gets all its bits flipped, in the final color vectors, x has different values in positions i, j, and y has the same values. Hence, x and y receive distinct colors. This completes the proof. $\square$

If $s \geq 2$, then the clique number of $H(p, s)$ is 2^{p-1} (see the comments after the proof of Theorem 3.3). Thus, $\chi(H(p, s)) = 2^{p-1}$, and so the above result is sharp.

6 Relationship Between Metric and Partition Dimension

In this section we investigate the relationship between metric and partition dimensions and how well they approximate each other. Our key concern is whether the parameters can be far apart. We first present several tools that will be used later.

A pair of vertices u,v is *redundant* if $N(u) - \{v\} = N(v) - \{u\}$. Clearly, if u and v are redundant and $w \neq u, v$, then $d(u, w) = d(v, w)$. Hence, if $\mathcal{P}$ is a resolving partition then no part can contain a redundant pair. Likewise, if S is a set of landmarks then $V(G) - S$ cannot contain a redundant pair. This leads to the following observations.

Observation 6.1. *In a resolving partition of the vertex set of a graph, no part contains a redundant pair.*

Observation 6.2. *If S is a set of landmarks in a graph, and u,v is a redundant pair, then S contains u or v.*

If U is a set of pendant vertices, each of which is adjacent to a common neighbor v, then the elements of U are pairwise redundant. Hence, the following.

Observation 6.3. *If a graph G contains a cut-vertex v and U is a set of k isolated vertices in $G - v$, then*

(i) given distinct u and v in U, no resolving partition has u and v in the same part, and

(ii) every set of landmarks contains at least $k - 1$ elements of U.

In [3] the following is shown.

Remark 6.4 (Chartrand, Salehi, Zhang 2000 [3]). For all natural numbers α and β where $\lceil \beta/2 \rceil + 1 \leq \alpha \leq \beta + 1$ there exists a graph G with $\dim_{\mathrm{P}}(G) = \alpha$ and $\dim_{\mathrm{M}}(G) = \beta$.

This led the authors of [3] to ask whether $\dim_{\mathrm{P}}(G)$ is always bounded below by $\lceil \dim_{\mathrm{M}}(G)/2 \rceil + 1$. We will see that the answer to this question is "no"; the bound in Remark 6.4 can be improved.

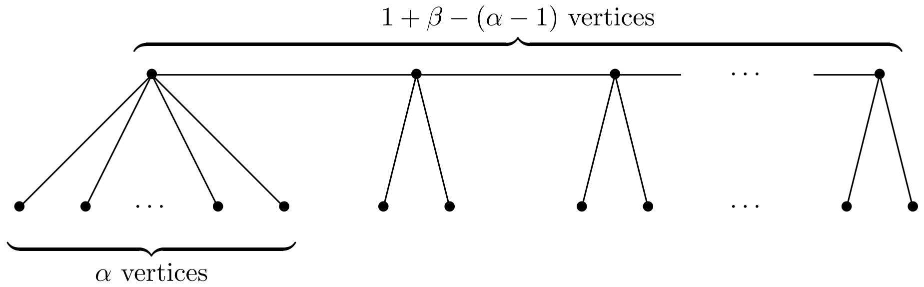

Figure 4: Example of the construction from Theorem 6.5: a graph with specified partition and metric dimensions.

Theorem 6.5. *Given natural numbers α and β where $3 \le \alpha \le \beta + 1$, there exists a graph G where $\dim_{\mathrm{P}}(G) = \alpha$ and $\dim_{\mathrm{M}}(G) = \beta$.*

Proof. If $\alpha = \beta + 1$ let G be a complete graph of order α. Note $\dim_{\mathrm{P}}(G) = \alpha$ and $\dim_{\mathrm{M}}(G) = \alpha - 1 = \beta$. So suppose $\alpha \le \beta$. Let Q be a path on $1+\beta-(\alpha-1)$ vertices. Attach α pendant vertices to one endpoint. Attach to each other vertex of Q exactly two pendant vertices, as in Figure 4. Call this caterpillar G. Let us call the end vertex adjacent to α pendants v. Since v is attached to α pendant vertices, we can use Observation 6.3 to see that $\dim_{\mathrm{P}}(G) \ge \alpha$. Suppose we put these α vertices into singleton sets $S_1, S_2, \ldots, S_\alpha$. Each other vertex of Q is attached to two pendant vertices. For each such pair, say x and y, put x in S_1, and y in S_2. Lastly, put all vertices of Q in S_1. As $\{S_1, S_2, \ldots, S_\alpha\}$ is a resolving partition we conclude that $\dim_{\mathrm{P}}(G) = \alpha$.

There are α isolated vertices in $G - v$. If we remove from G any vertex of Q other than v we produce two isolated vertices. Using Observation 6.3 we see that $\dim_{\mathrm{M}}(\mathrm{G}) \ge \beta$. Take the set S_2 described above and add to it all but one of the other pendant vertices adjacent with v. This set of cardinality β is resolving. Thus $\dim_{\mathrm{M}}(G) = \beta$. $\square$

By Remark 4.1, the 3 in the statement of Theorem 6.5 is best possible.

We now consider the order of graphs with small partition dimension and large metric dimension.

Remark 6.6. Let $\{\omega_n\}$ be an integer sequence that goes to infinity arbitrarily slowly, where $3 \le \omega_n < n$. Then there exists a sequence of graphs $\{G_n\}$ where

(i) G_n has order n,

(ii) $\dim_{\mathrm{P}}(G_n) = \omega_n$, and

(iii) $\dim_{\mathrm{M}}(G_n) = n - o(n)$.

Proof. For a given n, set $k = \lfloor (n-1)/\omega_n \rfloor$. Let $j = (n-1) - \omega_n k$. Let Q be the path whose vertices are $u_1, u_2, \ldots, u_k$, $v_1, v_2, \ldots, v_j$ (or just $u_1, u_2, \ldots, u_k$ if $j = 0$). To the vertex u_1 attach ω_n new pendant vertices. To each other vertex u_i ($i = 2, \ldots, k$) attach $\omega_n - 1$ new pendant vertices. Let G_n be the resulting caterpillar, and note that it has order n. In a manner similar to that of the proof of Theorem 6.5, one can show that $\dim_{\mathrm{P}}(G_n) = w_n$ and $\dim_{\mathrm{M}}(G_n) = n - 2k - j = n - o(n)$. $\square$

We might wonder whether there exists a sequence $\{G_n\}$ where $\dim_{\mathrm{P}}(G_n)$ is uniformly bounded and $\dim_{\mathrm{M}}(G_n) = n - o(n)$. Our next remark shows this is impossible.

Remark 6.7. If graph G has order n then

$$\dim_{\mathrm{M}}(G) \le \left(1 - \frac{1}{\dim_{\mathrm{P}}(G)}\right) n.$$

Proof. Suppose $\mathcal{P}$ is resolving partition of order p. Then some part of $\mathcal{P}$ contains at least n/p vertices. Let A be one of these parts. Note that $V(G) - A$ is a resolving set. $\square$

Complete graphs show that Remark 6.7 is sharp in some sense. However, for fixed partition dimension, the following improves the bound of Remark 6.7 for graphs of large order.

Theorem 6.8. *Let G be a graph of order n and partition dimension p. Then*

$$\dim_{\mathrm{M}}(G) \leq \left(1 - \frac{1}{p-1}\right) n + \frac{2^{p-1}}{p-1}.$$

Proof. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$ be a resolving partition of G. For each permutation σ of $\{1, 2, \ldots, p\}$ we construct a set $T_\sigma \subseteq V(G)$ as follows: Associate with every vertex $v \in V(G)$ the vector

$$\mathbf{r}(v, \sigma) := \left(d(v, P_{\sigma(1)}), d(v, P_{\sigma(2)}), \ldots, d(v, P_{\sigma(p)})\right).$$

Note that every such vector has a zero in exactly one position. Let T_σ be the set of all vertices u where $\mathbf{r}(u, \sigma)$ contains a one or two in a position to the right of its zero.

We proceed in two stages. We first verify that T_σ is a resolving set for G, for each σ. We then show that the mean, over all permutations σ, of $|T_\sigma|$ is at most the bound in the statement of this theorem. This will complete our proof.

Claim 1. For each σ, T_σ is a resolving set.

Let x and y be vertices of G. We need to show that there is some $v \in T_\sigma$ where $d(x, v) \neq d(y, v)$.

Suppose x and y are in the same part in $\mathcal{P}$. There must be another part that is closer to one, say x, than the other. If we consider a shortest path between x and this part, we will see it contains an element of T_σ and this element must be closer to x than y. To see this, note there must be some part $P_{\sigma(i)}$ in $\mathcal{P}$ such that $d(x, P_{\sigma(i)}) \neq d(y, P_{\sigma(i)})$. Without loss of generality, suppose $d(x, P_{\sigma(i)}) < d(y, P_{\sigma(i)})$. Select u from $P_{\sigma(i)}$ so that $d(x, u) = d(x, P_{\sigma(i)})$. Let Q be a shortest x,u-path. Let w be the vertex in Q adjacent to u. Select j so that $w \in P_{\sigma(j)}$. Note that $j \neq i$. If $\sigma(j) < \sigma(i)$, then $w \in T_\sigma$. If $\sigma(i) < \sigma(j)$ then $u \in T_\sigma$. Further, $d(x, u) \neq d(y, u)$, and $d(x, w) \neq d(y, w)$. Thus, T_σ differentiates x and y.

So, suppose x and y lie in different parts of $\mathcal{P}$. Without loss of generality, suppose the zero of $\mathbf{r}(x, \sigma)$ is to the right of the zero in $\mathbf{r}(y, \sigma)$. Consider $x = w_0, w_1, \ldots, w_k = y$, a shortest x,y-path. Choose i as large as possible so that the zero in $\mathbf{r}(w_i, \sigma)$ is to the right of the zero in $\mathbf{r}(y, \sigma)$. Suppose the kth entry of $\mathbf{r}(w_i, \sigma)$ is zero. Note the kth entry of $\mathbf{r}(w_{i+1}, \sigma)$ must be one, and, since this is to the right of the zero in $\mathbf{r}(w_{i+1}, \sigma)$, we note $w_{i+1} \in T_\sigma$.

If $w_{i+1} = y$ then, since $d(y, y) \neq d(x, y)$, T_σ differentiates x and y, and so we may suppose $w_{i+1} \neq y$. Consider w_{i+2}. The zeros in $\mathbf{r}(w_{i+2}, \sigma)$ and $\mathbf{r}(w_i, \sigma)$ must be in different positions, since otherwise this would contradict our assumption that i is the largest index for which w_i has the desired property. Thus, the position of the zero in $\mathbf{r}(w_i, \sigma)$ must contain either a one or two in $\mathbf{r}(w_{i+2}, \sigma)$. But this position comes to the right of the zero in $\mathbf{r}(w_{i+2}, \sigma)$. Thus, we see that $w_{i+2} \in T_\sigma$.

Thus, the minimum length x,y-path must contain two vertices, namely w_{i+1} and w_{i+2}, that lie in T_σ. One of these must lie at different distances from x and y. Hence, T_σ differentiates x and y and the proof of Claim 1 is complete.

Claim 2. The mean, over all σ, of $|T_\sigma|$ is at most

$$\left(1 - \frac{1}{p-1}\right) n + \frac{2^{p-1}}{p-1}.$$

Let A be the set of all vertices u where $d(u, P) \leq 2$ for all $P \in \mathcal{P}$. Suppose σ is chosen randomly from S_p, so that each permutation is chosen with equal probability. Let $v \in V(G)$. What is the probability that v lies in T_σ? If $v \in A$ then $v \in T_\sigma$ unless the zero in $\mathbf{r}(v, \sigma)$ lies in the last position. Thus, v lies in A with probability $1 - 1/p$. On the other hand, if $v \notin A$ then $\mathbf{r}(v, \sigma)$ has at most $p - 1$ positions with values in $\{0, 1, 2\}$, and so v lies in T_σ with probability at most $1 - 1/(p-1)$. We conclude the expected value of $|T_\sigma|$ is at most

$$
\begin{aligned}
\left(1-\frac{1}{p}\right)|A| \quad &+ \quad \left(1-\frac{1}{p-1}\right)(n-|A|) \\
&= \left(1-\frac{1}{p-1}\right)n + \left[\left(1-\frac{1}{p}\right) - \left(1-\frac{1}{p-1}\right)\right]|A| \\
&= \left(1-\frac{1}{p-1}\right)n + \left(\frac{1}{p(p-1)}\right)|A| \\
&\leq \left(1-\frac{1}{p-1}\right)n + \left(\frac{1}{p(p-1)}\right)2^{p-1}p \\
&= \left(1-\frac{1}{p-1}\right)n + \frac{2^{p-1}}{p-1}.
\end{aligned}
$$

This completes the proof. □

We now consider a construction that, for fixed $p \geq 3$, shows the bound in Theorem 6.8 is close to best possible. Let $k \geq 3$. We form sets of distinct vertices, say $S_0, S_1, \ldots, S_k$. Let S_0 be a singleton, and let each other set have cardinality $p-1$. Add edges so that each $S_i \cup S_{i+1}$ induces a complete graph. Let us call this graph K. Let n represent the order of K.

We observe the following:

(i) $\dim_{\mathrm{P}}(K) = p$,

(ii) $\dim_{\mathrm{M}}(K) = (p-2)k$, and

(iii) $n = (p-1)k + 1$.

Hence, $\dim_{\mathrm{M}}(K) = (1 - 1/(p-1))\,(n-1)$. We note that this expression, for fixed p, is $(1 - 1/(p-1))\,n + O(1)$, as is the formula in the statement of Theorem 6.8. This leads to the following conjecture.

Conjecture 6.9. *For every graph G of order n,*

$$\dim_{\mathrm{M}}(G) \leq \left(1 - \frac{1}{\dim_{\mathrm{P}}(G) - 1}\right) n + c,$$

where c is some positive constant.

It seems likely that the conjecture holds with $c = 2$.

7 Diameter

In [3] it is shown that if G has order n and diameter s then $n \leq (1+s)^{\dim_{\mathrm{P}}(G)}$. They ask if this bound is sharp. We answer this question by modifying their proof to produce a tighter bound.

Theorem 7.1. *If a graph G has order n, partition dimension p, and diameter s, then*

$$n \leq ps^{p-1}.$$

Proof. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$ be a resolving partition of minimum cardinality in a graph of diameter s. Label each vertex v with $\mathbf{r}(v|\mathcal{P})$. Since labels are distinct, there must be at least n different labels. Since v is in exactly one of the parts of $\mathcal{P}$, the vector $\mathbf{r}(v|\mathcal{P})$ contains exactly one zero. All other coordinates contain an integer from 1 to s. In placing a zero into a p-vector, there are p choices. Each of the remaining $p-1$ coordinates can be one of s different values. Applying the multiplication principle shows $n \leq ps^{p-1}$. □

Equality holds in the preceding theorem for complete graphs. Yet, equality holds for nothing else: Suppose equality holds for some p and some graph G of order n. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$ be a resolving partition of $V(G)$. Given a vector $\mathbf{w} = (x_1, x_2, \ldots, x_p)$ where each element is between zero and s, inclusive, and exactly one element is zero, there must be a vertex w for which $\mathbf{r}(w|\mathcal{P}) = \mathbf{w}$. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors that contain s in each coordinate, except $\mathbf{u}$ has a zero as its first coordinate and $\mathbf{v}$ has a zero as its second coordinate. Let u and v be vertices of G where $\mathbf{r}(u|\mathcal{P}) = \mathbf{u}$ and $\mathbf{r}(v|\mathcal{P}) = \mathbf{v}$. Let Q be a shortest u,v-path. Then P_1 contains at least s vertices of Q, as does P_2. Hence, Q has at least $2s$ vertices, and so $d(u, v) \geq 2s - 1$; G cannot have diameter s unless $s = 1$.

Finding general, sharp bounds on the order of a graph with given partition dimension and diameter may be difficult, but we can solve the diameter-two case.

First, we prove the following lemma (which we suspect may be well known). A family $\mathcal{F}$ of sets is *intersecting* if, for all A and B in $\mathcal{F}$, $A \cap B$ is nonempty.

Lemma 7.2. *Let $\mathcal{F}$ be an intersecting family of distinct subsets of a set of cardinality $n \geq 2$. If n is even, say $n = 2k$ for some integer k, then*

$$\sum_{A \in \mathcal{F}} |A| \leq k \left[\binom{2k-1}{k} + 2^{2k-1} \right].$$

Similarly, if $n = 2k + 1$, then

$$\sum_{A \in \mathcal{F}} |A| \leq (2k+1) \left[\binom{2k-1}{k} + 2^{2k-1} \right].$$

Further, both of these inequalities are sharp.

Proof. Let X be a set of cardinality $n \geq 2$. Let $\mathcal{F}$ be a family of subsets of X, containing all subsets of X with cardinality greater than $n/2$. If n is even, then, in addition, $\mathcal{F}$ contains each set of cardinality $n/2$ that includes some fixed $e \in X$.

Clearly, $\mathcal{F}$ is an intersecting family. Note that, for each $T \subseteq X$, $\mathcal{F}$ contains either T or $X - T$; further, if these have different sizes, then $\mathcal{F}$ contains the larger. Since no intersecting family can contain both T and $X - T$, we see that $\mathcal{F}$ attains the maximum value of $\sum_{A \in \mathcal{F}} |A|$, for an intersecting family.

It is not hard to verify that $\sum_{A \in \mathcal{F}} |A|$ is equal to the appropriate bound in the statement of the lemma. $\square$

Theorem 7.3. *The maximum order of a graph with diameter two and partition dimension $p \geq 2$ is*

$$k \left[\binom{2k-1}{k} + 2^{2k-1} \right], \quad \text{if } p = 2k,$$

and

$$(2k+1) \left[\binom{2k-1}{k} + 2^{2k-1} \right], \quad \text{if } p = 2k+1.$$

Proof. First we establish that the quantities in the statement of the theorem are upper bounds. Let G be a graph with diameter two and partition dimension p. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$ be a resolving partition. For each vertex v, let $S(v)$ be the set of all integers k ($1 \leq k \leq p$) for which the kth coordinate of $\mathbf{r}(v|\mathcal{P})$ does not equal two. Let $\mathcal{H} = \{\, S(v) : v \in G \,\}$. We will proceed by showing that $\mathcal{H}$ is an intersecting family and then applying Lemma 7.2.

Let u, v be vertices of G. Since G has diameter two, there exists a vertex w of G so that both u and v are either adjacent to, or equal to, w. The vector $\mathbf{r}(w|\mathcal{P})$ must have a zero in some coordinate. The vectors $\mathbf{r}(u|\mathcal{P})$ and $\mathbf{r}(v|\mathcal{P})$ cannot have a two in this same coordinate. Thus, this coordinate lies in both $S(u)$ and $S(v)$, and so these two sets have nonempty intersection. Hence $\mathcal{H}$ is intersecting. Now suppose $S \in \mathcal{H}$. We can bound the number of vertices u for which $S = S(u)$ by noting that there are $|S|$ places to put a zero in $\mathbf{r}(u|\mathcal{P})$ and $|S| - 1$ places for a one. All other entries of $\mathbf{r}(u|\mathcal{P})$ are two. Hence, there are at most $|S|$ vertices $v \in G$ with $S(v) = S$, and so the order of G is bounded above by $\sum_{S \in \mathcal{H}} |S|$. Thus, by Lemma 7.2 the order of G is bounded above by the quantity in the statement of the theorem.

It remains to show that the upper bounds we have established are sharp; we will use the graphs $H(p, s)$ defined in Example 2.1. Let $n = p$, and let $\mathcal{F}$ be an intersecting family as described in the proof of Lemma 7.2. Based on $\mathcal{F}$, we now construct a subgraph G of $H(p, 2)$ having the desired properties. For a vertex $v \in H(p, 2)$, let $T(v)$ be the set of coordinates of v that are less than two. Given $A \in \mathcal{F}$, let $T^{-1}(A) = \{v : T(v) = A\}$. Note that $|A| = |T^{-1}(A)|$, since there are $|A|$ choices for where to put the zero coordinate. Let G be the subgraph of $H(p, 2)$ induced by $\bigcup_{A \in \mathcal{F}} T^{-1}(A)$. Then $|V(G)| = \sum_{A \in \mathcal{F}} |T^{-1}(A)| = \sum_{A \in \mathcal{F}} |A|$, giving the desired order by Lemma 7.2.

To see that $\dim_{\mathrm{P}}(G) = p$, note first that $\mathcal{F}$ is closed under taking supersets, and so G is a natural subgraph of $H(p, 2)$; thus $\dim_{\mathrm{P}}(G) \leq p$, by Observation 2.3. If $\dim_{\mathrm{P}}(G)$ were strictly less than p, then the order of G would be too small, by the first part of this proof; thus $\dim_{\mathrm{P}}(G) = p$. To see that G has diameter two, select vertices u and v from G. Suppose $u \in T^{-1}(A)$ and $v \in T^{-1}(B)$. Let j be a common element of A and B. Let w be the vertex of $H(p, 2)$ that contains only ones, except a zero in the jth coordinate. Then u and v are both a distance of at most one away from w. Hence, G has diameter at most two. $\square$

Determining the maximum order of a graph with given partition dimension and diameter appears to be more difficult when the diameter exceeds two.

Problem 7.4. *Determine, for each p and s, the maximum order of a graph having diameter s and partition dimension p.*

We conclude with a construction of graphs with fixed diameter and partition dimension, and large order. Given natural numbers p and s, we construct a subgraph $K(p, s)$ of $H(p, s)$ (defined in Example 2.1). Let V be those vertices in $H(p, s)$ with either zero or one in their first position. There are s^{p-1} vertices having zero in their first position and $(p-1)s^{p-2}$ vertices having one in their first position, since, in such a vertex, there are $p - 1$ possible places for the zero. Hence, $|V| = s^{p-1} + (p-1)s^{p-2}$. Let $K(p, s)$ be the subgraph of $H(p, s)$ induced by V. We note that $K(p, s)$ has diameter s. Further, $K(p, s)$ is a natural subgraph of $H(p, s)$ and thus has partition dimension at most p, by Observation 2.3. Counting the number of vertices in $K(p, s)$ gives the following.

Remark 7.5. For given p and s, there exists a graph of order $(s + p - 1)s^{p-2}$ having diameter s and partition dimension p.

By Theorem 7.1 and Remark 7.5, we have the following partial answer to Problem 7.4.

Corollary 7.6. *Given p and s, the maximum order of a graph with partition dimension p and diameter s is between $(s + p - 1)s^{p-2}$ and ps^{p-1}, inclusive.*

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