COMPUTING THE DERIVATIVE OF $x^{2/3}$ FROM THE DEFINITION: A HARDER-THAN-NECESSARY EXAMPLE

First, I should have remembered that I needed to use

$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right).$$

Speaking informally, to compute the limit of a fraction below, we start with the "$a-b$" in the numerator of the fraction. In particular, $a = (x+h)^{2/3}$ and $b = x^{2/3}$ in the calculation below. Then we multiply by "a^2+ab+b^2" to get "a^3-b^3". Note $a^3 = (x+h)^2$ and $b^3 = x^2$ in this case.

So, if $f(x) = x^{2/3}$ then, by the definition,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^{2/3} - x^{2/3}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^{2/3} - x^{2/3}}{h}\ \frac{(x+h)^{4/3} + (x+h)^{2/3}x^{2/3} + x^{4/3}}{(x+h)^{4/3} + (x+h)^{2/3}x^{2/3} + x^{4/3}} \\
&= \lim_{h\to 0} \frac{\left((x+h)^{2/3}\right)^3 - \left(x^{2/3}\right)^3}{h\ \left[(x+h)^{4/3} + (x+h)^{2/3}x^{2/3} + x^{4/3}\right]} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h\ \left[(x+h)^{4/3} + (x+h)^{2/3}x^{2/3} + x^{4/3}\right]} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h\ \left[(x+h)^{4/3} + (x+h)^{2/3}x^{2/3} + x^{4/3}\right]} \\
&= \lim_{h\to 0} \frac{2x + h}{(x+h)^{4/3} + (x+h)^{2/3}x^{2/3} + x^{4/3}} \\
&= \frac{2x + 0}{(x+0)^{4/3} + (x+0)^{2/3}x^{2/3} + x^{4/3}} \\
&= \frac{2x}{x^{4/3} + x^{4/3} + x^{4/3}} = \frac{2x}{3x^{4/3}} \\
&= \frac{2}{3}\, x^{-1/3}.
\end{aligned}$$

It will be a lot easier to remember the power law *stated in section 3.1.*

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