The Riemann-Lebesgue Lemma

Lemma. *If $f(x)$ is piecewise continuous on $[-\pi, \pi]$ then*

$$\lim_{m\to\infty} \int_{-\pi}^{\pi} f(x) \cos mx \, dx = 0$$

and

$$\lim_{m\to\infty} \int_{-\pi}^{\pi} f(x) \sin mx \, dx = 0.$$

Proof. Fourier's choice as an ortho*normal* set is

$$\{\phi_n(x)\}_{n=0}^{\infty} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin 2x, \frac{1}{\sqrt{\pi}} \cos 2x, \dots \right\}.$$

Given an $f(x)$ which is piecewise continuous, let us compare $f(x)$ to *any* finite linear combination from this orthonormal set. That is, we form a finite linear combination

$$g(x) = \sum_{n=0}^{N} c_n \phi_n(x)$$

and we compare by finding the area of the squared difference between $f(x)$ and $g(x)$:

$$E := \int_{-\pi}^{\pi} (f(x) - g(x))^2 \, dx.$$

We can understand this integral. First,

$$E \overset{\circledast}{=} \int_{-\pi}^{\pi} f^2(x) \, dx - 2 \int_{-\pi}^{\pi} f(x) g(x) \, dx + \int_{-\pi}^{\pi} g^2(x) \, dx.$$

The last integral simplifies by orthonormality:

$$\begin{aligned}
\int_{-\pi}^{\pi} g^2(x) \, dx &= \int_{-\pi}^{\pi} \left(\sum_{n=0}^{N} c_n \phi_n(x) \right) \left(\sum_{m=0}^{N} c_m \phi_m(x) \right) dx \\
&= \sum_{n=0}^{N} \sum_{m=0}^{N} c_n c_m \int_{-\pi}^{\pi} \phi_n(x) \phi_m(x) \, dx = \sum_{n=0}^{N} \sum_{m=0}^{N} c_n c_m \delta_{nm} \\
&= \sum_{n=0}^{N} c_n^2.
\end{aligned}$$

The middle integral relates to the coefficients we expect to use. Let

$$A_n = \int_{-\pi}^{\pi} f(x) \, \phi_n(x) \, dx.$$

Then

$$\int_{-\pi}^{\pi} f(x) g(x) \, dx = \sum_{n=0}^{N} c_n \int_{-\pi}^{\pi} f(x) \, \phi_n(x) \, dx = \sum_{n=0}^{N} c_n A_n.$$

On the other hand, from the definition of E we see (very clearly!) that $E \geq 0$. From $\circledast$ we can therefore say

$$\int_{-\pi}^{\pi} f^2(x)\,dx - 2\sum_{n=0}^{N} c_n A_n + \sum_{n=0}^{N} c_n^2 \geq 0.$$

Now for a trick of a low class, algebraic, type: We can complete the squares to rewrite the two sums in this last inequality. In particular

$$-2\sum_{n=0}^{N} c_n A_n + \sum_{n=0}^{N} c_n^2 = \sum_{n=0}^{N}(c_n^2 - 2c_n A_n + A_n^2) - \sum_{n=0}^{N} A_n^2 = \sum_{n=0}^{N}(c_n - A_n)^2 - \sum_{n=0}^{N} A_n^2,$$

Therefore

$$\int_{-\pi}^{\pi} f^2(x)\,dx + \sum_{n=0}^{N}(c_n - A_n)^2 - \sum_{n=0}^{N} A_n^2 \geq 0. \tag{1}$$

Inequality (1) applies for *any* linear combination $g(x) = \sum_{n=0}^{N} c_n \phi_n(x)$. In particular, if we choose c_n to be equal to A_n we get

$$\int_{-\pi}^{\pi} f^2(x)\,dx - \sum_{n=0}^{N} A_n^2 \geq 0.$$

Rewriting very slightly, this is *Bessel's inequality*,

$$\sum_{n=0}^{N} A_n^2 \leq \int_{-\pi}^{\pi} f^2(x)\,dx. \tag{2}$$

Finally we get to prove the lemma! By Bessel's inequality, the list of coefficients A_n must decrease to zero, because (2) applies for any N but the right side of (2) is just some positive number independent of N. That is,

$$\lim_{N\to\infty} A_N = 0.$$

On the other hand, if $n = 2m$ is even we have

$$A_{2m} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

while if $n = 2m-1$ is odd we have

$$A_{2m-1} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

We are done. □

Remark. *The decision to compare $f(x)$ to an arbitrary finite linear combination from the orthonormal set is not obvious and requires insight. On the other hand, in our abstract view of the space $L^2(-\pi,\pi)$ of functions as having "Euclidean geometry" the decision to do the comparison by using a sum of squares is actually reasonably natural. We are finding the length of the "vector" $f - g$ in the function space.*