Final Exam

Total of 150 points

DUE at NOON on *Wednesday 19 December, 2007*

Rules. You *may* use written or online references for basic facts including trigonometric identities and integrals, but you should indicate clearly when doing so. You *may* use MATLAB and other calculation technology to produce plots and compute numbers.

You *may not* seek out complete solutions by searching online, in textbooks, or elsewhere. You *may not* talk to, or communicate with by any method, any person other than *me* about the content of this exam.

Please come to me with questions, including questions about these rules. I will be genuinely helpful and not obscure. Or email `ffelb@uaf.edu`.

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1. (*15 pts*) Recall that "Fourier's choice" of an orthogonal set is

(1)
$$\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \dots\}$$

in the space of functions $L^2(-\pi, \pi)$. (This space contains functions which are defined on $(-\pi, \pi)$ and have finite integral $\int_{-\pi}^{\pi} |f(x)|^2 \, dx$.) The classical Fourier series is

(2)
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

One finds the coefficients by the formulas

(3)
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

The Question: Generalize the orthogonal set (1) and the formulas (2) and (3) to an arbitrary length interval $(-c, c)$ by the change of variables $y = cx/\pi$. *Do* give the details.

2. (*25 pts*) Solve the following heat-in-a-rod problem by separation of variables, noting that the length of the rod is 5:

$$
\begin{aligned}
PDE \qquad\qquad & u_t = 3u_{xx}, \\
BCs \qquad\qquad & u(0, t) = 0, \qquad u(5, t) = 0, \\
IC \qquad\qquad & u(x, 0) = 2x,
\end{aligned}
$$

Although such a problem should be very standard by now, *do* start with a search for the separated solutions, *do* find the eigenvalues and eigenfunctions, *do* write down the general solution to the PDE and BCs, and finally *do* find the coefficients in the series expansion of the initial condition and thus the coefficients in the solution itself.

3. (*25 pts*) Solve this boundary value problem for the Laplace equation by separation of variables, on the rectangle $0 < x < 1$, $0 < y < \pi$, to find $u(x, y)$:

$$PDE \qquad\qquad u_{xx} + u_{yy} = 0,$$

$$u(x, \pi) = 0$$

$$BCs \qquad u(0, y) = 0 \quad\boxed{}\quad u(1, y) = 0$$

$$u(x, 0) = 1$$

4. Start by reading Lesson 21 in FARLOW. Now consider the PDE and boundary conditions describing the displacement $u(x, t)$ of a uniform elastic beam of length 1 which is rigidly held at $x = 0$ but moves freely at the $x = 1$ end:

$$PDE \qquad\qquad u_{tt} = -u_{xxxx},$$

$$BCs \qquad \begin{aligned} u(0, t) &= 0, \\ u_x(0, t) &= 0, \\ u_{xx}(1, t) &= 0, \\ u_{xxx}(1, t) &= 0. \end{aligned}$$

FARLOW calls this problem the "cantilever-beam", with sketch on page 166.

(a) (*5 pts*) Consider separated solutions $u(x, t) = T(t)X(x)$ where $T(t) = a\sin(\omega t) + b\cos(\omega t)$. Show that $X'''' - \omega^2 X = 0$.

(b) (*10 pts*) Show that, from the BCs and the PDE, the eigenvalues ω solve the equation

$$\left(\cos\sqrt{\omega} + \cosh\sqrt{\omega}\right)^2 + \sin^2\sqrt{\omega} - \sinh^2\sqrt{\omega} = 0.$$

Reduce this equation through trigonometric identities to

$$\cos\sqrt{\omega} + \left(\cosh\sqrt{\omega}\right)^{-1} = 0.$$

(*Hint*: The matrix problem $A\mathbf{x} = 0$ has a nontrivial solution if and only if $\det(A) = 0$.)

(c) (*10 pts*) Actually, in the presence of gravity the horizontally-oriented beam deflects downward in a curved shape at equilibrium. Its displacement $\tilde{u}(x)$ solves

$$ODE \qquad\qquad 0 = -\alpha^2 \tilde{u}'''' - \rho g,$$

$$BCs \qquad\qquad \tilde{u}(0) = 0, \quad \tilde{u}'(0) = 0, \quad \tilde{u}''(1) = 0, \quad \tilde{u}'''(1) = 0$$

where $\rho > 0$ is a constant linear density and $g > 0$ is the gravity constant. Find $\tilde{u}(x)$. (*Hint*: This linear homogeneous ODE has solutions which are polynomial in x and can be found by repeated integration.)

Comment: The PDE BVP stated at the beginning of this problem is for displacement *from equilibrium*. That is, $u(x,t)$ describes how far the beam deflects from the equilibrium position $\tilde{u}(x)$. The true position of the beam is $\tilde{u}(x) + u(x,t)$.

(d) *Extra Credit.* Find numerical approximations to the first three eigenvalues ω solving the equation in part **(b)**. Show your answers to four digits past the decimal point. (*Note.* I get $\omega_1 = 3.52$, $\omega_2 = 22.03$, $\omega_3 = 61.70$, displayed to two digits.)

5. **(a)** (*10 pts*) Solve this wave equation initial value problem on the whole real line *using the Fourier transform on the spatial variable x*:

$$PDE \qquad\qquad u_{tt} = c^2 u_{xx},$$

$$ICs \qquad\qquad \begin{aligned} u(x,0) &= f(x), \\ u_t(x,0) &= 0. \end{aligned}$$

Stop when you have a formula for $u(x,t)$ which is a double integral involving $f(x)$.

(b) (*5 pts*) Although no boundary conditions are stated above, and they seem not to be needed when using the Fourier transform, I noted in class that implicitly we need some "boundary conditions at infinity". Describe what conditions we need for the computation in **(a)** and why.

(c) (*10 pts*) Continue from the solution in part **(a)** by changing order of integration and writing

$$u(x,t) = \int_{-\infty}^{\infty} W(t, y-x) f(y)\, dy.$$

Give an integral formula for W, which I would call the "wave kernel." Address the convergence of the integral which defines W.

(d) (*5 pts*) Write down D'Alembert's solution for the problem stated in part **(a)**.

(e) *Extra Credit.* Derive the solution in part **(d)** from the solution in part **(c)**.

> All of the above problems are required. Now choose *one* of the following two problems.

I. When electrical signals move along a wire they travel similarly to waves in a vibrating string. For long wires, even ones made from good electrical conductors, one observes that the signals are "washed out". This issue became a huge barrier to the first construction of an Atlantic telegraph cable in the 1850s. Kelvin solved the problem by applying the methods you have learned in Math 421. He made himself and many others very wealthy,

in the first really big *technological* IPO (Initial Public Offering) ever, but that's a longer story ...

An equation describing the voltage $u(x,t)$ in a wire is the PDE

$$(4) \qquad\qquad u_{tt} + \gamma u_t = c^2 u_{xx}$$

for $\gamma \geq 0$. (This equation, in slightly greater generality, it is derived from laws for electrical circuits at

$$\texttt{www.math.ubc.ca/\textasciitilde feldman/apps/telegrph.pdf}$$

You *may* go online for that if you want, but it is not essential to this problem.) Consider the concrete "vibrating string" type of problem for PDE (4), on a wire of length π, using these conditions:

$$BCs \qquad\qquad \begin{aligned} u_x(0,t) &= 0, \\ u_x(\pi,t) &= 0, \end{aligned}$$

$$ICs \qquad\qquad \begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= 0. \end{aligned}$$

(a) (*15 pts*) Solve this problem by separation of variables. Note that the ODE for $T(t)$ in the separated solutions is not completely familiar, but you can solve it by the usual methods. The soolution to this part will be an infinite sum for $u(x,t)$. (You may assume that c and γ are such that the case $\gamma = 2cn$ never occurs for any integer n. *Additional Hint*: You may want to use the functions cosh and sinh in describing $T(t)$ in some cases.)

(b) (*10 pts*) Carefully check that your answer in **(a)** solves the vibrating string equation (wave equation) when $\gamma = 0$, as it should.

(c) (*5 pts*) Find the steady state assuming $\gamma > 0$. Is there a steady state when $\gamma = 0$? Explain.

(d) *Extra Credit.* Make as much progress as you can in describing the heat equation as a limiting case of equation (4). Explain how your solution from part **(a)** becomes a solution to the heat equation in this limiting case.

II. *Another problem Kelvin solved is to estimate the age of the earth. His method used the Fourier transform as explained in the handout from the book by Körner. Here is an alternate method.*

We model the earth as a rod with length $2L$ where L is the radius of the earth. Heat is lost at each end of the rod but the sides are insulated. We assume this "earth" started at a uniform temperature θ_0 at some time in the distant past, where θ_0 is the melting point of rock. This problem is symmetric around the middle of the rod. Assuming reasonably

that the temperature is a smooth function of x, the spatial derivative of the temperature at the midpoint of the rod (of length $2L$) is therefore always zero.

Let's assume the ends of the rod—the surface of our "earth"—have temperature zero, where zero is the average temperature of the atmosphere, which is controlled by the sun. By the symmetry mentioned above, we have this problem for the temperature $u(x, t)$:

$$PDE \qquad\qquad u_t = Ku_{xx},$$

$$BCs \qquad\qquad \begin{aligned} u(0, t) &= 0, \\ u_x(L, t) &= 0, \end{aligned}$$

$$IC \qquad\qquad u(x, 0) = \theta_0,$$

Note that the three constants so far, L, K, and θ_0 are all experimentally measurable if we assume the earth is made of a single kind of rock, for instance.

(a) *(5 pts)* Sketch the rod as described in the paragraph starting with "We model ...". In words, explain why this rod model can also be regarded as modeling the earth as an infinite slab of rock of thickness $2L$. (This is a *finite-thickness flat earth* model.)

(b) *(15 pts)* Solve the PDE IBVP by separation of variables.

(c) *(10 pts)* Now for the crucial additional data. By measuring temperatures in mines, Kelvin knew roughly what the heat flux rate was at the surface of the earth. That is, he also measured the positive value

$$\Phi_0 = +u_x(0, t)$$

where t is the current time. That is, he measured the surface geothermal flux rate at time t which is the amount of time which has passed since the earth was molten ("$u(x, 0) = \theta_0$"), the earth's age. Therefore differentiate your solution from **(b)** to write "$\Phi_0 = -u_x(0, t)$" as

$$(5) \qquad\qquad \Phi_0 = F(Kt)$$

for a function F which is defined by an infinite series. Argue that the infinite series which defines F converges for any value $Kt > 0$. Solving (5) for t give the age of the earth.

There is much to criticize in Kelvin's calculation, but he was essentially right. Modern geophysics routinely uses comparable arguments based, usually, on many more physical observations and more complete physical theories.

(d) *Extra Credit.* Assuming all constants are 1: $L = 1$, $\theta_0 = 1$, $K = 1$, $\Phi_0 = 1$, solve $\Phi_0 = F(Kt)$ numerically to find t, the age of the earth. *Extra Extra Credit.* Find and use physically reasonable values for the constants; solve for t. *Extra^3 Credit.* Add radioactive heating and find t again.