Assignment #10

DUE *Wednesday 5 December, 2007*

Note: "HANDOUT" refers to the copied pages 84–99 of Chapter 4 of Brown and Churchill, *Fourier Series and Boundary Value Problems.* The exercises are on pages 95–97 of the HANDOUT.

"Dirichlet's Theorem", as I refer to it in class and below, is the Corollary on page 92. Note that you will use it in answering exercises **3**, **5**, and **6**(a) in the HANDOUT.

Exercise 1. Exercise **3** on HANDOUT. The Fourier series in question is

$$f(x) \sim \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}.$$

But, and this is important to understanding the question, you do not need to *know* the Fourier series to answer this question! (That is, you do not need to know the coefficients in the Fourier series.)

Exercise 2. Exercise **5** on HANDOUT.

Exercise 3. Exercise **6** (a) on HANDOUT.

Exercise 4. Exercise **9** on HANDOUT.

Exercise 5. Exercise **10** on HANDOUT.

Exercise 6. Let's remember why we are interested in Fourier series. Consider this heat in a rod problem, with a rod of length π:

$$\begin{array}{ll} PDE & u_t = u_{xx}, \\ BCs & \begin{array}{l} u_x(0,t) = 0, \\ u_x(\pi,t) = 0, \end{array} \\ IC & u(x,0) = \phi(x), \end{array}$$

(a) Apply separation of variables in the usual way. As a result, you will write a formula for the solution $u(x,t)$ as a Fourier cosine series, with integral formulas for the coefficients. (*There is no reason to go into all the gory details here. We have seen them before many* *times. But write down enough to remind yourself, and to demonstrate to me, how the story goes!*)

(b) Now assume

$$\phi(x) = \begin{cases} -1, & 0 \le x < \pi/2, \\ 0, & x = \pi/2, \\ 1, & \pi/2 < x \le \pi. \end{cases}$$

Find the Fourier cosine series for $u(x,t)$ and for $u(x,0)$ in particular. Sketch plots of the $t = 0$, small t, large t, and steady state cases.

(c) Apply Dirichlet's Theorem to the even periodic extension of $\phi(x)$ to show that your solution $u(x,t)$ actually converges to $\phi(x)$ when $t = 0$.

(d) On the other hand, in what ways do partial sums of the Fourier cosine series for $\phi(x) = u(x,0)$ look different from $\phi(x)$? Explain why the partial sums of the series for $u(x,t)$, *for small* $t > 0$, are excellent approximations of $\phi(x)$.

Exercise 7. In this problem, assume $f(x)$ is defined on $[-\pi, \pi]$, is an *even* function, and is *differentiable* with a continuous derivative.

(a) I showed in class that because $f(x)$ is even it has a cosine series (using "Fourier's choice" for an orthogonal set):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

Show that the coefficients a_n can be found by integrating over half the interval:

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx.$$

(b) Argue that $f'(x)$ is odd. (*You can, and probably should, do this without reference to Fourier series ideas. Imagine yourself as a calculus student again* ...)

(c) Do integration by parts on the integral

$$\int_0^{\pi} f(x) \cos nx \, dx$$

to find a formula for the coefficients a_n by integrating a product of $f'(x)$ and something.

(d) Argue that if $f'(x)$ has a convergent Fourier sine series then the Fourier cosine series for $f(x)$ converges faster than the Fourier series for general $f(x)$. (*This is an instance of the general rule that if a function is more "regular", meaning it is smooth because you can differentiate it, then its Fourier series is better behaved.*)