Midterm Exam # 2: SOLUTIONS *(to practice version)*

1. (a)
$$f'(t) = \frac{1}{2}t^{-1/2} + \frac{1}{2}t^{-3/2}.$$
(b)
$$f'(x) = e^{x^2}\left(2x^4 + 7x^2 + 2\right).$$
(c)
$$y' = \frac{1}{\ln 10}\,\frac{2x}{x^2+1}.$$
(d)
$$y' = \frac{(\cos x)(e^x + \cot x) - (\sin x)(e^x - \csc^2 x)}{(e^x + \cot x)^2}$$
and simplifications thereof.
(e)
$$h'(t) = \frac{1}{t} + \frac{t}{t^2+1} - \frac{1}{t-1}.$$

2. (a) The tangent line is $y + 1 = 0(x - \pi)$ or $y = -1$.
(b) First, by implicit differentiation,
$$\frac{dy}{dx} = \frac{-2x - y}{x + 4y^3},$$
so tangent line is $y - 1 = -\frac{5}{6}(x-2)$ or $y = -\frac{5}{6}x + \frac{8}{3}$.

3. Write $y = \arccos x$ as $\cos y = x$. Apply implicit differentiation to this, to get
$$-\sin y \frac{dy}{dx} = 1$$
or
$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}.$$

4. (a) The velocity function is $v(t) = s'(t) = 3t^2 - 9t - 7$. We want to find t when $v(t) = 5$, so we solve
$$3t^2 - 9t - 7 = 5 \quad \text{or} \quad 3t^2 - 9t - 12 = 0 \quad \text{or} \quad 3(t-4)(t+1) = 0.$$
The problem says the position function is only meaningful for $t \geq 0$ so we ignor $t = -1$. Thus the velocity is 5 when $t = 4$ seconds.
(b) The acceleration switches from negative to positive when it goes through zero. On the other hand, the acceleration function is $a(t) = s''(t) = 6t - 9$. Thus we solve $a(t) = 0$ or $6t - 9 = 0$ to get $t = 3/2$ seconds. (To check that the particle switches from decelerating to accelerating, note $a(0) < 0$ and $a(2) > 0$.)

5. An example is shown in Figure 1. Note that the graph is always concave up but is decreasing on $(-\infty, -2)$ and $(2, \infty)$, while it is increasing on $(-2, 2)$.

6. We apply the closed interval method, that is, we check both the critical numbers and the endpoints. Note
$$f'(x) = \frac{1-x^2}{(x^2+1)^2} \quad \text{so we solve} \quad 1 - x^2 = 0 \quad \text{to get} \quad x = \pm 1.$$

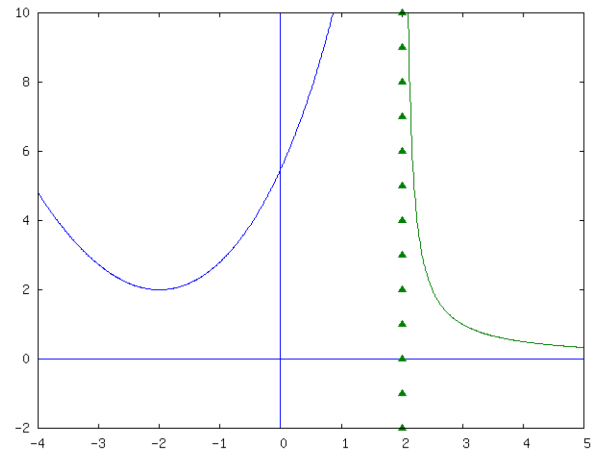

FIGURE 1. A solution to # 5.

We are only interested in the critical number $x = +1$ because it is on the interval $[0, 5]$. Therefore we check this one critical number and the two end points:

x	y
0	0
1	$\frac{1}{2}$
5	$\frac{5}{26} \approx \frac{1}{5}$

The absolute maximum of $1/2$ is attained at $x = 1$; abs min of 0 at $x = 0$.

7. Let x be the distance from the bottom of the ladder to the wall. Let y be the height up the wall of the top of the ladder. Assuming the wall is vertical and perpendicular to the floor, $x^2+y^2 = 12^2$. Notice x and y are functions of t. The rates of change are related by the equation coming from differentiating the Pythagorean equation:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

We want dy/dt at the time when $x = 6$. At this same time, $y = \sqrt{12^2 - 6^2} = \sqrt{108} = 6\sqrt{3}$. Note that $dx/dt = +1$ at any time; that's the speed at which the base of the ladder is sliding away from the wall. Thus

$$\frac{dy}{dt} = -\frac{x\frac{dx}{dt}}{y} = -\frac{6 \cdot 1}{6\sqrt{3}} = -\frac{1}{\sqrt{3}}.$$

The speed itself is $+1/\sqrt{3}$; note $dy/dt < 0$ because the ladder is sliding down.

8. Let's suppose the curve is the graph of a function $y(x)$. The information we have says $y(0) = 5$ and

$$\frac{dy}{dx} = 2y.$$

This equation says that $y(x)$ is the exponential

$$y(x) = A\,e^{2x}.$$

But $5 = y(0) = Ae^0 = A$ so

$$y(x) = 5\,e^{2x}.$$

Extra Credit. Several possibilities, but most possibilities are waves with sharp points. For example: A sawtooth wave, with maximum of 1 at $\ldots, -4, -2, 0, 2, 4, \ldots$ and minimum of -1 at $\ldots, -3, -1, 1, 3, 5, \ldots$ and sharp points at each maximum and minimum.