MATH 200 (Bueler) Fall 2008

Midterm Exam # 2: SOLUTIONS

1. (a) $f'(x) = \sec x \, \tan x + x^{-1}$

(b)

$$\frac{dy}{dx} = \frac{x^2 - 8x - 4}{(x-4)^2}$$

(c) Implicit differentiation:

$$\begin{aligned} e^y \frac{dy}{dx} \sin x + e^y \cos x &= -\sin y \frac{dy}{dx} \\ \frac{dy}{dx} \left[e^y \sin x + \sin y \right] &= -e^y \cos x \\ \frac{dy}{dx} = \frac{-e^y \cos x}{e^y \sin x + \sin y} &= -\frac{\cos x}{\sin x + e^{-y} \sin y} \end{aligned}$$

(d)

$$g'(s) = \frac{1}{\ln 2} \frac{-3}{1-3s} = \frac{3}{(\ln 2)(3s-1)}$$

2. (a)

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

(b) Implicit differentiation:

$$\begin{aligned} y &= \arctan x \\ \tan y &= x \\ \sec^2 y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} \end{aligned}$$

(Recall the trigonometric identity $\sec^2 \theta = 1 + \tan^2 \theta$ which follows from $1 = \cos^2 \theta + \sin^2 \theta$ by dividing by $\cos^2 \theta$.)

3.

$$\frac{dy}{dx} = \cos(\sin x)\, \cos x \qquad \text{therefore} \qquad m = \left.\frac{dy}{dx}\right|_{x=\pi} = \cos(0)\cos(\pi) = -1$$

so the line is $y - 0 = (-1)(x - \pi)$ or $y = \pi - x$.

4. (a) $s'(t)$ always exists; $s'(t) = 4t^3 - 4t = 4t(t-1)(t+1)$; critical numbers where $s'(t) = 0$ i.e. $t = 0, -1, +1$

(b) build the table at right; conclude absolute max is at $t = 2$ and $s = 10$ while absolute min of $s = 1$ occurs twice, at $t = -1$ and at $t = 1$

t	$s(t)$
-1	1
0	2
1	1
2	10

5. Surface area of a sphere is $S = 4\pi r^2 = 4\pi(d/2)^2 = \pi d^2$ if d is the diameter. Thus

$$\frac{dS}{dt} = 2\pi\, d\, \frac{dd}{dt}$$

Since $dS/dt = -1\,\mathrm{cm}^2/\mathrm{min}$, and when $d = 8$ cm,

$$-1 = 2\pi\,(8)\,\frac{dd}{dt}$$

Thus

$$\frac{dd}{dt} = \frac{-1}{16\pi} \quad \frac{\mathrm{cm}}{\mathrm{min}}$$

6. (a) if t is measured in years,

$$m(t) = 1000\left(\frac{1}{2}\right)^{t/30} \qquad \mathrm{mg}$$

(b)

$$\begin{aligned} 1 &= 1000\left(\frac{1}{2}\right)^{t/30} \\ \frac{t}{30} &= \log_{1/2}\left(10^{-3}\right) = \frac{3\ln 10}{\ln 2} \\ t &= \frac{90\ln 10}{\ln 2}\ \text{years} \end{aligned} \qquad (\approx 299\,\text{years, from a calculator})$$

7. *Many correct solutions. One possible graph is increasing the whole way and is continuous, but "levels out" to slope zero at $x = 2$ and at $x = 4$.*

8. (a)

$$L(x) = f(a) + f'(a)(x-a) = e^0 + e^0(x-0) = 1 + x$$

(b)

$$e^{0.03} = f(0.03) \approx L(0.03) = 1 + 0.03 = 1.03$$

(*By a calculator,* $e^{0.03} = 1.030454$, *so this is correct to three digits past the decimal point.*)
(c) *Not shown.*

Extra credit. Let $p(x)$ be the polynomial. Suppose there are four or more distinct real roots. Let x_1, x_2, x_3, x_4 be four distinct roots listed in increasing order. By applying Rolle's theorem on $[x_1, x_2]$, $[x_2, x_3]$, $[x_3, x_4]$, we find distinct values c_1, c_2, c_3 at which $p'(c_1) = p'(c_2) = p'(c_3) = 0$. But $p'(x)$ is a quadratic polynomial which has at most two roots (of any kind). This contradiction shows p has at most 3 real roots.