Answers to SAMPLE Final Exam

1.

$$\int_5^7 f(x)\,dx = \int_0^7 f(x)\,dx - \int_0^5 f(x)\,dx = 6 - 10 = -4$$

2.

$$f'(x) = x^3 - 2x + C_1 = x^3 - 2x - 3 \qquad \text{because } 1 = f'(2) = 8 - 4 + C_1 \text{ so } C_1 = -3$$

$$f(x) = \frac{x^4}{4} - x^2 - 3x + C_2 = \frac{x^4}{4} - x^2 - 3x + 6 \qquad \text{because } 0 = f(2) = 4 - 4 - 6 + C_2 \text{ so } C_2 = 6$$

In conclusion,

$$f(x) = \frac{x^4}{4} - x^2 - 3x + 6$$

3. **(a)** For an equally-spaced grid and right endpoints,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

and

$$x_i = 0 + i\Delta x = \frac{2i}{n}.$$

Therefore

$$\int_0^2 3x + 4\,dx = \lim_{n\to\infty} \sum_{i=1}^n (3x_i + 4)\ \Delta x = \lim_{n\to\infty} \sum_{i=1}^n \left(3\left(\frac{2i}{n}\right) + 4\right)\ \frac{2}{n}$$

(b)

$$\sum_{i=1}^n \left(3\left(\frac{2i}{n}\right) + 4\right)\ \frac{2}{n} = \frac{2}{n}\left(\frac{6}{n}\sum_{i=1}^n i + \sum_{i=1}^n 4\right) = \frac{2}{n}\left(\frac{6}{n}\frac{n(n+1)}{2} + 4n\right)$$

$$= \frac{2}{n}\left(3(n+1) + 4n\right) = \frac{14n+6}{n}$$

so

$$\int_0^2 3x + 4\,dx = \lim_{n\to\infty} \frac{14n+6}{n} = 14$$

(c)

$$\int_0^2 3x + 4\,dx = \frac{3x^2}{2} + 4x\Big]_0^2 = \frac{12}{2} + 8 = 14$$

4. **(a)** using $u = x^3 + 1$ and converting limits to u-values,

$$\int_0^1 x^2 \sec^2\left(x^3+1\right)\ dx = \frac{1}{3}\int_1^2 \sec^2 u\,du = \frac{1}{3}\tan u\Big]_1^2 = \frac{1}{3}\left(\tan 2 - \tan 1\right)$$

(b) using $u = 5\theta$ and recalling an integral,

$$\int \tan(5\theta)\,d\theta = \frac{1}{5}\int \tan u\,du = -\frac{1}{5}\ln|\cos u| + C = -\frac{1}{5}\ln|\cos(5\theta)| + C$$

(c) using $u = \sin x$,

$$\int \frac{\cos x}{\sin^4 x}\,dx = \int \frac{du}{u^4} = \int u^{-4}\,du = -\frac{1}{3}u^{-3} + C = -\frac{1}{3}\left(\sin x\right)^{-3} + C$$

5. use the Second FTC and the chain rule:

$$f'(x) = \frac{d}{dx}\left(\int_3^{x^2} \cos(e^t)\,dt\right) = 2x\cos\left(e^{x^2}\right)$$

6. using $u = \sqrt{x}$ so $x = u^2$ and $2u\,du = dx$,

$$\int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2u\,du}{u\,(1+u^2)} = 2\int \frac{du}{1+u^2} = 2\arctan u + C = 2\arctan(\sqrt{x}) + C$$

7.

$$\begin{aligned}\int_1^5 \ln x\,dx &\approx \frac{b-a}{3n}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)\\ &= \frac{4}{12}\left(\ln 1 + 4\ln 2 + 2\ln 3 + 4\ln 4 + \ln 5\right)\\ &= \frac{1}{3}\left(\ln 16 + \ln 9 + \ln 64 + \ln 5\right) = \frac{1}{3}\ln(46080)\end{aligned}$$

[*any of the last three forms is fine*]

EC. Factor the numerator $x^3 = x \cdot x^2$. Then use $u = 1 - x^2$, so $x^2 = 1 - u^2$ and $x\,dx = -du/2$:

$$\begin{aligned}\int \frac{x^3}{\sqrt{1-x^2}}\,dx &= -\frac{1}{2}\int \frac{(1-u)}{\sqrt{u}}\,du = \frac{1}{2}\int u^{1/2} - u^{-1/2}\,du\\ &= \frac{1}{3}u^{3/2} - u^{1/2} + C = \frac{1}{3}(1-x^2)^{3/2} - (1-x^2)^{1/2} + C\end{aligned}$$

8. If the two numbers are x and y then $x + y = 18$ and we want to maximimize $P = xy$. Note $0 \le x \le 18$ because both x and y are positive. Solving $x + y = 18$ for y and substituting gives

$$P(x) = x(18 - x) = -x^2 + 18x.$$

The maximum is at the critical number because $P(0) = 0$ and $P(18) = 0$ at the endpoints. So

$$P'(x) = -2x + 18 = 0 \qquad \text{or} \qquad x = 9.$$

The numbers are 9 and 9. (*The sum is 18 and the product is 81. Compare pairs (8.9,9.1) or (8,10) or (7,11), etc.*)

9. **(a)** $y = 1$ is a horizontal asymptote and both $x = +1$ and $x = -1$ are vertical asymptotes

(b) first

$$f'(x) = \frac{(2x)(x^2-1) - (x^2-4)(2x)}{(x^2-1)^2} = \frac{6x}{(x^2-1)^2}$$

so $f(x)$ is decreasing on $(-\infty,-1)\bigcup(-1,0)$ and increasing on $(0,1)\bigcup(1,\infty)$

(c) A good sketch shows the asymptotes as dashed lines and has "x" and "y" labels on the axes. The x-intercepts are $(-2,0)$ and $(2,0)$, the y-intercept is $(0,4)$, and these are all labeled. The graph is even (i.e. symmetric across the y-axis). The solid curve actually approaches the asymptotes and is increasing/decreasing as in **(b)**.

10. **(a)** $f'(x) = -2x^{-3}$ so $f'(5) = -2/5^3$

(b)

$$\begin{aligned} f'(5) &= \lim_{\Delta x \to 0} \frac{f(5+\Delta x) - f(5)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{(5+\Delta x)^2} - \frac{1}{5^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{5^2 - (5+\Delta x)^2}{(5+\Delta x)^2 5^2 \Delta x} \\ &= \lim_{\Delta x \to 0} \frac{5^2 - 5^2 - 10\Delta x - \Delta x^2}{(5+\Delta x)^2 5^2 \Delta x} = \lim_{\Delta x \to 0} \frac{-(10+\Delta x)\Delta x}{(5+\Delta x)^2 5^2 \Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-(10+\Delta x)}{(5+\Delta x)^2 5^2} = -\frac{10}{5^4} = -\frac{2}{5^3}. \end{aligned}$$

11. **(a)**

$$\frac{dy}{dx} = 2\left(\frac{1}{2}\right) x^{-1/2} + 6e^x = x^{-1/2} + 6e^x$$

(b)

$$\int 2\sqrt{x} + 6e^x \, dx = 2\left(\frac{2}{3}\right) x^{3/2} + 6e^x + C = \frac{4}{3}x^{3/2} + 6e^x + C$$

12. Find dy/dx by implicit differentiation:

$$\begin{aligned} 2x - \left(1 \cdot y + x \cdot \frac{dy}{dx}\right) + 3y^2 \frac{dy}{dx} &= 0 \\ \frac{dy}{dx}\Big[-x + 3y^2\Big] &= -2x + y \\ \frac{dy}{dx} &= \frac{2x - y}{x - 3y^2} \end{aligned}$$

The slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{(2,0)} = \frac{4}{2} = 2$$

so the equation of the tangent line is

$$y - 0 = 2(x-2) \qquad \text{or} \qquad y = 2x - 4.$$

13. There are two similar triangles: one triangle is formed by the streetlight and the ground out to the tip of the man's shadow and the other triange is formed by the man and his shadow on the ground. Let x be the length of the man's shadow and let y be the distance of the man from the base of the streetlight. Then by similar triangles we have

$$\frac{x}{6} = \frac{x+y}{15} \qquad \text{or} \qquad 15x = 6x + 6y \qquad \text{or} \qquad x = \frac{2}{3}y$$

Because x and y are functions of t (i.e. they are changing), we have

$$\frac{dx}{dt} = \frac{2}{3}\frac{dy}{dt}$$

But

$$\frac{dy}{dt} = 3 \text{ ft/sec}$$

so his shadow is lengthening at

$$\frac{dx}{dt} = \left(\frac{2}{3}\right) 3 = 2 \text{ ft/sec}$$

(*We never used the information that the man is 10 feet from the base of the light! But the rate at which his shadow is lengthening turns out to be constant. It is proportional to the rate at which he is walking, which is constant.*)

14. Either

the values of $f(x)$ can be made as close as you wish to L by choosing values of x sufficiently close to, but not equal to, c

or

for all $\epsilon > 0$ there exists $\delta > 0$ so that if $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$

15. *This has to do with inverse functions and implicit differentiation.* Let

$$y = \arcsin x.$$

This is equivalent to

$$\sin y = x$$

So the derivative with respect to x is

$$\cos y \, \frac{dy}{dx} = 1$$

or

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

We want to write the right side in terms of x, and we know $\sin y = x$. But by the trig. identity $\cos^2 \theta + \sin^2 \theta = 1$ we have

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

But then

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

which is what we wanted to show.