Fourier Transform from Fourier Series

This turns out to be more subtle than I remembered. These comments are an incomplete version of the full story.

Recall that

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} \right\}_{n=-\infty}^{\infty}$$

is an orthonormal set in the space of functions $L^2(-\pi,\pi)$, on which the inner product is $(f,g) = \int_{-\pi}^{\pi} \overline{f(x)} g(x)\, dx$. Therefore if a function has complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \frac{1}{\sqrt{2\pi}} e^{inx} \tag{1}$$

it follows that the coefficients are

$$a_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx. \tag{2}$$

On the other hand, a slight modification of Dirichlet's theorem shows that if $f(x)$ is a function on $[-\pi,\pi]$ which is piecewise smooth then the right side of equation (1) actually converges to $f(x)$.

In some sense, equations (1) and (2) are a kind of transformation/(inverse transformation) pair. One can go "forwards" from $f(x)$ to its Fourier series coefficients a_n and then "back" to $f(x)$.

As noted in class, equations (1) and (2) are reasonably analogous; in both cases there is a limit of a sum of infinitely many numbers times complex exponentials. (The "numbers" are a_n in (1) and $f(x)$ in (2).) But (1) is a discrete sum, not an integral. On the other hand, discrete sums are connected to integrals, *by definition*.

Definition. *(*The equally-spaced case of Riemann's definition of the integral.*) If $f(x)$ is continuous on $[a,b]$ then*

$$\int_a^b f(x)\, dx := \lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\Delta x$$

where

$$\Delta x = \frac{b-a}{N}$$

and x_n is any number in the interval $[a_n, b_n]$ where $a_n = a + (n-1)\Delta x$ and $b_n = a_n + \Delta x$.

Note in particular that the terms in a sum have to have a coefficient like "Δx", namely something which is proportional to $1/N$ where N is the number of terms in the sum, if the sum is to have a limit which is an integral. We have to be adding up the areas of thin rectangles which get thinner at the right rate if we are to have an integral.

So we proceed to try to rewrite the Fourier series pair (1), (2) as two integrals. In the first step we replace the interval $[-\pi,\pi]$ with an arbitrary length interval $[-c,c]$. The second step is to take a limit so that interval becomes $(-\infty,\infty)$. The third step is to use the definition of the Riemann integral to replace the sum in (1) with an integral.

<u>FIRST STEP.</u> Replace x in the integral in (2) by $y = cx/\pi$; get

$$\frac{c}{\pi} a_n = \frac{1}{\sqrt{2\pi}} \int_{-c}^{c} e^{-in\pi y/c} f\left(\frac{\pi y}{c}\right)\, dy.$$

Define these rescalings

$$\omega_n := \frac{n\pi}{c} \qquad \text{and} \qquad g(y) := f\left(\frac{\pi y}{c}\right) \qquad \text{and} \qquad G(\omega_n) := \frac{c}{\pi}a_n. \tag{3}$$

Note that $g(y)$ is defined for $y \in [-c, c]$ but $G(\omega)$ is only defined for a discrete list of inputs ω, namely $\omega_n = n\pi/c$ where n is an integer. This list of allowed inputs becomes denser and denser in the real line as $c \to \infty$, however.

We get a rescaling of (1),

$$g(y) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{i\omega_n y} G(\omega_n)\, \frac{\pi}{c}, \tag{4}$$

and of (2) to an arbitrary interval,

$$G(\omega_n) = \frac{1}{\sqrt{2\pi}} \int_{-c}^{c} e^{-i\omega_n y} g(y)\, dy. \tag{5}$$

The sum in (4) is written to look somewhat more like a Riemann sum than the sum in (1), but we are not there yet.

<u>Second Step.</u> We take the limit as $c \to \infty$ in formulas (4) and (5) to get

$$g(y) = \frac{1}{\sqrt{2\pi}} \lim_{c\to\infty} \sum_{n=-\infty}^{\infty} e^{i\omega_n y} G(\omega_n)\, \frac{\pi}{c}, \tag{6}$$

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega y} g(y)\, dy. \tag{7}$$

Note that $g(y)$ is defined on the whole real line when $c \to \infty$ because $g(y) = f(\pi y/c)$. Also, note formula (7) actually serves as a definition of $G(\omega)$ for any ω.

<u>Third Step.</u> *This step is the subtle one.* The remaining issue is to understand (6) as a Riemann sum. There are two limit processes in (6) because the sum needs to become an integral *and* the integral needs to be on an infinite interval.

But we have to be careful. Rewrite (6) as

$$g(y) = \frac{1}{\sqrt{2\pi}} \lim_{c\to\infty} \lim_{M\to\infty} \sum_{n=-M}^{M} e^{i\omega_n y} G(\omega_n)\, \frac{\pi}{c}.$$

Both c and M get arbitrarily large, but to proceed we take the double limit *assuming* $c = M$. Then we can see the extra coefficient "π/c" as a "$\Delta\omega$",

$$\frac{\pi}{c} = \frac{2\pi c}{2M^2} =: \Delta\omega,$$

for an integral on an interval $[-\pi c, \pi c]$ with length $2\pi c$. Now we see $\omega_n = n\pi/c = n\Delta\omega$. We have a Riemann sum:

$$g(y) = \frac{1}{\sqrt{2\pi}} \lim_{c\to\infty} \lim_{M\to\infty} \sum_{n=-M}^{M} e^{i\omega_n y} G(\omega_n)\, \Delta\omega = \frac{1}{\sqrt{2\pi}} \lim_{c\to\infty} \int_{-\pi c}^{\pi c} e^{i\omega y} G(\omega)\, d\omega.$$

Finally we recognize the improper integral. Taking the last limit $c \to \infty$ and recalling (7) we have the Fourier transform pair:

$$g(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega y} G(\omega)\, d\omega, \qquad G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega y} g(y)\, dy,$$

that is,

$$g(y) = \mathcal{F}^{-1}[G](y), \qquad G(\omega) = \mathcal{F}[g](\omega).$$