Solutions to Exam on *DGC&AT*

1.

$$\nabla\cdot(\nabla\times\mathbf{F}) = \nabla\cdot\left((\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z})\,\hat{\mathbf{i}} - (\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z})\,\hat{\mathbf{j}} + (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})\,\hat{\mathbf{k}} \right)$$
$$= \frac{\partial^2 F_z}{\partial x\partial y} - \frac{\partial^2 F_y}{\partial z\partial x} - \frac{\partial^2 F_z}{\partial x\partial y} + \frac{\partial^2 F_x}{\partial y\partial z} + \frac{\partial^2 F_y}{\partial z\partial x} - \frac{\partial^2 F_x}{\partial y\partial z} = 0$$

assuming mixed partial derivatives commute.

2. The easiest way is to convert to cylindrical coordinates first:

$$\mathbf{F} = (x^2+y^2)^{-3}(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}) = (r^2)^{-3}(r\,\hat{\mathbf{r}}) = r^{-5}\,\hat{\mathbf{r}}.$$

Here r and $\hat{\mathbf{r}}$ are the cylindrical coordinates formulas.

Then, using the standard formula,

$$\nabla\cdot\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}\left(r(r^{-5})\right) = r^{-1}(-4)r^{-5} = -\frac{4}{r^6}.$$

In cartesian coordinates, $\nabla\cdot\mathbf{F} = -4(x^2+y^2)^{-3}$; one can also get this formula by differentiating using the cartesian formula for the divergence.

3 (a). Because $\nabla\times\mathbf{F} = \mathbf{0}$, and because this fact is true on all of $\mathbb{R}^3$, a simply-connected region, we conclude that $\mathbf{F}$ is path-independent. In particular, a Φ can be defined so that $\nabla\Phi = \mathbf{F}$. A formula for such a Φ is

$$\Phi(x,y,z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds.$$

This formula is only interesting because it reminds us that we need path independence for this to be a valid definition.

But we also know $\nabla\cdot\mathbf{F} = 0$. Thus

$$\nabla^2\Phi = \nabla\cdot(\nabla\Phi) = \nabla\cdot\mathbf{F} = 0.$$

Extra Credit [3 (b)]. This is genuinely extra credit in that we have not yet addressed *how* to do this. But *what* to do is clear: find a scalar function Φ defined on all of $\mathbb{R}^3$ with the property

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0,$$

that is, Laplace's equation $\nabla^2\Phi = 0$. Furthermore we want Φ to *not* just be constant or linear, because we want $\mathbf{F} = \nabla\Phi$ to be non-constant. Here is such a Φ:

$$\Phi(x,y,z) = x^2 - y^2.$$

Then

$$\mathbf{F} = 2x\,\hat{\mathbf{i}} - 2y\,\hat{\mathbf{j}}.$$

You may check that this is a curl-free and a divergence-free vector field.

We will have various ways of producing such formulas in this semester and next semester. In Math 422 we will learn that the real and imaginary part of a nice function of one complex variable $z = x + iy$ will satisfy Laplace's equation. And

$$x^2 - y^2 = \Re(z^2).$$

4. For a single charge at the origin, clearly the distribution of charge is spherically symmetric. Thus the electric field $\mathbf{E}$ is spherically symmetric (so it only depends on r) and points only in the radial direction: $\mathbf{E} = E(r)\,\hat{\mathbf{r}}$. Thus by Gauss' law, if S is a sphere of radius R centered at the origin,

$$\frac{q}{\epsilon_0} = \iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,dS = \iint_S E(R)\,\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\,dS = E(R)\iint_S dS = E(R)4\pi R^2.$$

This holds for any $R > 0$. It follows that, as claimed by Coulomb,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}.$$

5. The definition we want is that $\nabla\times\mathbf{F}$ is a vector at each point p for which

$$\hat{\mathbf{n}}\cdot(\nabla\times\mathbf{F}) = \lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds$$

where $\hat{\mathbf{n}}$ is any unit vector based at p and where C is a small closed curve in a plane through p with normal $\hat{\mathbf{n}}$ enclosing a surface of area S, and where the orientation of C is such that $\hat{\mathbf{n}}$ and C are right-hand oriented. (*Note that explaining how C, ΔS, and $\hat{\mathbf{n}}$ are related is essential for this to be a definition.*)

The picture is not shown here. See me if desired.

6. By the divergence theorem applied to $\mathbf{F} = \mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$,

$$\frac{1}{3}\iint_S \mathbf{r}\cdot\hat{\mathbf{n}}\,dS = \frac{1}{3}\iiint_V \nabla\cdot\mathbf{r}\,dV = \frac{1}{3}\iiint_V (1+1+1)\,dV = \iiint_V dV = V.$$

7. This would be a huge pain except for one fact:

$$\nabla\cdot\mathbf{F} = (2xy) + (-2xy) + 0 = 0.$$

Thus $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_V \nabla\cdot\mathbf{F}\,dV = 0$ by the divergence theorem.

8. We know a certain fact for every sphere *centered at the origin.* On the other hand, the definition of the divergence of a vector field $\mathbf{F}$ at a point p is the limit

$$\nabla\cdot\mathbf{F}\Big|_p = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS.$$

The limit is taken over surfaces S around p enclosing volumes ΔV.

Thus we know enough to compute the divergence at the origin:

$$\nabla\cdot\mathbf{F}\Big|_{(0,0,0)} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \lim_{R\to 0}\frac{1}{(4/3)\pi R^3}\,\pi R^3 = \lim_{R\to 0}\frac{3}{4} = \frac{3}{4}.$$

Here "S" stands for the sphere of radius R centered at the origin. Recall that for such a sphere the volume is $\Delta V = (4/3)\pi R^3$.

Thus we know that the divergence *at the origin* is $3/4$. We do not know the divergence elsewhere. We do not know that the divergence is constant.