

Math 302 – Exam 1 Review

February 27, 2001

1. **Verifying Solutions:** You should be able to verify that a given function satisfies an initial value problem. Example: $y = xe^x$ satisfies $y'' - 2y' + y = 0$ with $y(0) = 0$, $y'(0) = 1$. To verify a solution, compute y' and y'' and substitute them in to the differential equation to see that it is satisfied and then check that the initial conditions also hold.

2. **Existence and Uniqueness Theorem:** The theorem says that the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$ has a unique solution (that is, a solution exists and is unique) if f and $\frac{\partial f}{\partial y}$ are continuous on some rectangle containing (t_0, y_0) . If f and $\frac{\partial f}{\partial y}$ are NOT continuous near (t_0, y_0) (in particular AT (t_0, y_0)) then anything could happen: there may be no solution, one solution or many solutions.

Example: $y' = te^y$ will have unique solutions for any initial conditions, while $y' = \frac{t}{y}$ may not have a unique solution to the initial value problem $y(t_0) = 0$, since f is not continuous at $y = 0$.

3. **First Order Linear Equations:** Equations of the form $y' + p(x)y = f(x)$ can be solved by multiplying both sides by the integrating factor $\mu = e^{\int p(x)dx}$, which then yields the equation $\frac{d}{dx}(\mu y) = \mu f(x)$ which is then integrated to yield $\mu y = \int \mu f(x)dx$ or $y = \frac{1}{\mu} \left(C + \int \mu f(x)dx \right)$ or $y = e^{-\int p(x)dx} \left(C + \int f(x)e^{\int p(x)dx} dx \right)$. I think it is better to memorize the method rather than the above formula.

4. **Separation of Variables:** If a first order DE can be written in the form $N(y)dy = M(x)dx$ then we can obtain a family of implicit solutions of the form

$$\int N(y)dy = \int M(x)dx + C$$

Sometimes we won't find all of the solutions, but this will give an infinite family of solutions.

5. **Mixing Problems:** Note: Upon further review, I have decided to reinstate this type of problem for the exam. If a substance is being added and removed from a solution then the amount of substance in the solution, $Q(t)$ is governed by the equation

$$\frac{dQ}{dt} = \text{RATE IN} - \text{RATE OUT}$$

For nice examples we can solve this DE to find the amount of substance in the solution at any time t .

6. **Exact Equations:** A differential equation $Mdx + Ndy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

In this case we can find a function of two variable ψ so that $\frac{\partial \psi}{\partial x} = M$ and $\frac{\partial \psi}{\partial y} = N$ and solutions to the DE are given by $\psi(x, y) = c$ where c is a constant. To find ψ we compute: $\psi = \int Mdx + g(y)$ where $g(y)$ is some function to be determined by the equation $\frac{\partial \psi}{\partial y} = N$.

Example: $3x^2y^2dx + (2x^3y + e^y)dy$ can be shown to be exact and so

$$\psi = \int 3x^2y^2dx + g(y) = x^3y^2 + g(y)$$

Next we differentiate by y to get $N = 2x^3y^2 + g'(y)$ and hence $g'(y) = e^y$ or $g(y) = e^y$ and hence $\psi = x^3y^2 + e^y$ and the solution to the DE is $x^3y^2 + e^y = c$.

7. Direction Fields and autonomous equations: You should be able to sketch a direction field. Given an autonomous equation, such as $y' = y^3 - 4y$, you should be able to find the equilibrium solutions, determine whether they are stable or unstable and be able to sketch solutions for various initial conditions.

8. Linear Second Order DE: You should know the general form of a second order linear differential equation: $y'' + p(t)y' + q(t)y = g(t)$ and what the corresponding homogeneous equation is. A fundamental set of solutions of a homogeneous second order linear differential equation is a set of two linearly independent functions y_1 and y_2 which both solve the equation $y'' + p(x)y' + q(x)y = 0$ and whose Wronskian is non-zero. The Wronskian is

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

y_1 and y_2 are a fundamental system precisely when this Wronskian is non-zero. Every solution to the homogeneous equation is of the form $y = c_1y_1 + c_2y_2$ for some c_1 and c_2 .

9. 2nd Order with Constant Coefficients: I expect you to be able to solve any eqn of the form $ay'' + by' + cy = 0$, and to interpret the results, that is, are the solutions periodic, do they tend to zero or infinity as $t \rightarrow \infty$, etc.

10. Undetermined Coefficients: You need to understand theorems 3.6.1 and 1.6.2 be able to use undetermined coefficients to solve an equation.

Formulas

I will provide the formulas:

$$\mu_x = \frac{M_y - N_x}{N} \mu \quad \mu_y = \frac{N_x - M_y}{M} \mu$$

which are useful in finding integrating factors for non-exact equations.