Poisson's equation by the FEM, continued: general boundary conditions and error analysis

This note extends [2]. It assumes the reader has absorbed chapter one of [5].

Consider the boundary value problem described in the classical manner as

$$(1) \qquad -\triangle u = f \quad \text{on } D \subset \mathbb{R}^2, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \text{and} \qquad \frac{\partial u}{\partial n} = g_N \quad \text{on } \Gamma_N$$

where ∂D is the boundary of an open, connected region D. Assume that the subsets $\Gamma_D, \Gamma_N \subset \partial D$ are disjoint, $\Gamma_D \cup \Gamma_N = \partial D$, and $\frac{\partial u}{\partial n}$ is the directional derivative of u in the outward direction normal to ∂D. Suppose ∂D is a smooth or piecewise-smooth closed, continuous curve. Furthermore, assume f, g_D, g_N are well-behaved enough so that we do not worry about them; concretely, suppose Γ_N is open, f is continuous on D and g_D, g_N are continuous and bounded on Γ_D, Γ_N respectively.

Now we seek a variational formulation of (1). Let v be a function in a (for now) unspecified space. Multiply $f = -\triangle u$ by v and integrate:

$$\int_D fv = -\int_D v\triangle u = \int_D \nabla v \cdot \nabla u - \int_{\partial D} v\frac{\partial u}{\partial n} = \int_D \nabla v \cdot \nabla u - \int_{\Gamma_D} v\frac{\partial u}{\partial n} - \int_{\Gamma_N} vg_N,$$

by Green's theorem. Of the two boundary terms on the right, the second is clearly desirable because it brings g_N into the variational formulation. On the other hand, and based on the pure Dirichlet problem for (1) addressed in [2], we assume v is zero on Γ_D in order to eliminate that boundary term.

I now introduce not-completely-standard, but good(!) notation because we need to talk about functions which are specified on a piece of the boundary of D.

Definition. *Suppose $\Gamma \subset \partial D$ is measurable. Let*

$$H^1_\varphi(\Gamma) = \left\{ v \text{ measurable on } D \,\middle|\, v \in L_2(D),\ \nabla v \in L_2(D),\ v = \varphi \text{ on } \Gamma \right\}.$$

Comparing to existing notation, $H^1_0(D)$ becomes $H^1_0(\partial D)$ in this notation. Note $H^1_\varphi(\Gamma) \subset H^1(D)$ is a vector subspace only if $\varphi = 0$. If $\varphi \neq 0$ then it is an *affine* space with $v - w \in H^1(\Gamma)$ if $v, w \in H^1_\varphi(\Gamma)$. Finally, in this definition, as in all of [5] as well, the mathematical issue of *traces* is ignored. A "trace operator" is the precise way to define boundary values of a $H^1(D)$ function; see [3, section 5.5] or [1, section 1.6].

The *variational formulation* of (1) is to find $u \in H^1_{g_D}(\Gamma_D)$ such that

$$(2) \qquad \int_D \nabla u \cdot \nabla v = \int_D f\, v + \int_{\Gamma_N} g_N\, v, \qquad \text{for all } v \in H^1_0(\Gamma_D).$$

That is, we require u to take on the desired values g_D on Γ_D while test functions v are zero on Γ_D. If $\Gamma_D = \emptyset$ then a solution to (1) or (2) is not unique as an arbitrary constant can be added to any solution to get a new one.

Before formulating a finite element version of (2), we make two additional assumptions on the geometry of the domain and its boundaries. First, we require that D is in fact a *polygonal* domain with vertices (corners) q_s, $s = 1, \ldots, M$. Secondly, we suppose Γ_D is a union of closed edges of ∂D, that is, the "transition" from Γ_D to Γ_N occurs only at vertices of the polygon D. These assumptions facilitate error analysis in the finite element method.

What is the finite element method here, how accurate is it, and how to compute it in practice? We restrict ourselves to the simplest case of piecewise linear finite elements on a triangulation. Suppose the triangulation $\mathcal{T}$ on D has nodes p_j for $j = 1, \ldots, N_p$. Furthermore, suppose the set of nodes $\{p_j\}$ includes all the vertices q_s. Also, suppose that the first N nodes ($N \leq N_p$) are either in the interior of D or are in Γ_N and that nodes $p_{N+1}, \ldots, p_{N_p}$ are in Γ_D. The values of u at the nodes $p_1, \ldots, p_N$ are the unknowns.

Let V_h be the space of continuous functions which are linear on each $T \in \mathcal{T}$ and which are zero on Γ_D, so that

$$V_h \subset H_0^1(\Gamma_D).$$

For $j = 1, \ldots, N_p$, let φ_j be the "hat" function on p_j, that is, φ_j is continuous, piecewise linear, and satisfies $\varphi_j(p_k) = \delta_{jk}$. For $j = 1, \ldots, N$, $\varphi_j \in V_h$. Furthermore, $\{\varphi_j\}_{j=1}^N$ is a basis of V_h.

The *finite element formulation* is to seek real coefficients ξ_j, $j = 1, \ldots, N$, for the approximate solution u_h,

$$u_h = \sum_{j=1}^{N} \xi_j \varphi_j + \sum_{j=N+1}^{N_p} g_D(p_j) \varphi_j, \tag{3}$$

so that

$$\int_D \nabla u_h \cdot \nabla v = \int_D f v + \int_{\Gamma_N} g_N v \qquad \text{for all } v \in V_h,$$

that is, so that

$$\int_D \nabla u_h \cdot \nabla \varphi_k = \int_D f\, \varphi_k + \int_{\Gamma_N} g_N\, \varphi_k, \qquad \text{for all } k = 1, \ldots, N. \tag{4}$$

Note that $u_h \in H^1(D)$ and $u_h\big|_{\Gamma_D}$ is a piecewise linear approximation to g_D along Γ_D, and thus "$u_h \in H^1_{g_D}(\Gamma_D)$" generally only in an approximate sense.

It follows, by substitution of (3) into (4), that we will solve a linear system $A\mathbf{x} = \mathbf{b}$ where

$$a_{kj} = \sum_{j=1}^{N} \int_D \nabla \varphi_j \cdot \nabla \varphi_k \quad \text{and} \quad b_k = \int_D f\, \varphi_k + \int_{\Gamma_N} g_N\, \varphi_k - \sum_{j=N+1}^{N_p} g_D(p_j) \int_D \nabla \varphi_j \cdot \nabla \varphi_k.$$

Note the presence of stiffness terms on the right side.[1]

[1] One may also formulate the problem with additional trivial equations so as to have all stiffness terms on the left-hand-side.

Though existence is not a particular concern here, A is symmetric and positive definite. In fact, if $w = \sum_{j=1}^{N} y_j \varphi_j \in V_h$ is nonzero and at least one point $p_j \in \Gamma_D$ then $0 < \int_D |\nabla w|^2 = \mathbf{y}^\top A\, \mathbf{y}$ where $\mathbf{y} = (y_j)$. Thus we have unique existence for the *discrete* problem (3) and (4).

Code. Here is an implementation which barely fits on one page. As in [2], the integrals $\int_D f\, \varphi_k$ are computed by first approximating f by a piecewise linear function. Similarly, the boundary integrals $\int_{\Gamma_N} g_N\, \varphi_k$ are done by Simpson's rule on each edge of the triangulation which is in Γ_N. Note the inclusion of a helper function `Ncntrb` which computes Neumann boundary contributions to the load vector. Many further details of the implementation are addressed for the pure Dirichlet case ($\Gamma_N = \emptyset$, $g_D \equiv 0$) in [2].

```
function [uh,un]=poissonDN(f,gD,gN,fd,fGam,h0,p,t)
%POISSONDN  Solve Poisson's equation on (open) domain D by the FE method ...

geps=.001*h0;  int=(feval(fd,p) < -geps);    % int true if node in interior
inGamD=(feval(fGam,p) < +geps)&(~int);       % inGamD true if node in GamD
inGamN=~(int|inGamD);                        % inGamN true if node in GamN
if sum(inGamD)==0, error('no unique soln; needs Dirichlet points'), end
Np=size(p,1);  uh=zeros(Np,1);  ff=uh;  un=uh;  % Np=total # of nodes
N=sum(~inGamD);  un(~inGamD)=(1:N)';           % N=# of unknowns
for j=1:Np % eval f once for each node; fill in known bdry vals on Gam1
    ff(j)=feval(f,p(j,:));
    if inGamD(j), uh(j)=feval(gD,p(j,:)); end, end

% loop over triangles to set up stiffness matrix A and load vector b
A=sparse(N,N);  b=zeros(N,1);
for n=1:size(t,1)
    j=t(n,1);  k=t(n,2);  l=t(n,3);  vj=un(j);  vk=un(k);  vl=un(l);
    J=[p(k,1)-p(j,1), p(l,1)-p(j,1); p(k,2)-p(j,2), p(l,2)-p(j,2)];
    ar=abs(det(J))/2;  C=ar/12;  Q=inv(J'*J);  fT=[ff(j) ff(k) ff(l)];
    % go through nodes and compute stiffness and Dirichlet contribution
    if vj>0
        A(vj,vj)=A(vj,vj)+ar*sum(sum(Q)); b(vj)=b(vj)+C*fT*[2 1 1]';
        if vk>0
            A(vj,vk)=A(vj,vk)-ar*sum(Q(:,1));  A(vk,vj)=A(vj,vk); end
        if vl>0
            A(vj,vl)=A(vj,vl)-ar*sum(Q(:,2));  A(vl,vj)=A(vj,vl); end
    else % pj in GamD
        if vk>0, b(vk)=b(vk)+uh(j)*ar*sum(Q(:,1)); end
        if vl>0, b(vl)=b(vl)+uh(j)*ar*sum(Q(:,2)); end, end
    if vk>0
        A(vk,vk)=A(vk,vk)+ar*Q(1,1);  b(vk)=b(vk)+C*fT*[1 2 1]';
        if vl>0
            A(vk,vl)=A(vk,vl)+ar*Q(1,2);  A(vl,vk)=A(vk,vl); end
    else % pk in GamD
        if vj>0, b(vj)=b(vj)+uh(k)*ar*sum(Q(:,1)); end
        if vl>0, b(vl)=b(vl)-uh(k)*ar*Q(1,2); end, end
    if vl>0
        A(vl,vl)=A(vl,vl)+ar*Q(2,2);  b(vl)=b(vl)+C*fT*[1 1 2]';
    else % pl in Gam1
```

```
            if vj>0, b(vj)=b(vj)+uh(l)*ar*sum(Q(:,2)); end
            if vk>0, b(vk)=b(vk)-uh(l)*ar*Q(1,2); end, end
        % now add Neumann contribution
        if inGamN(j)
            if ~int(k), b(vj)=b(vj)+Ncntrb(gN,p(j,:),p(k,:)); end
            if ~int(l), b(vj)=b(vj)+Ncntrb(gN,p(j,:),p(l,:)); end,  end
        if inGamN(k)
            if ~int(j), b(vk)=b(vk)+Ncntrb(gN,p(k,:),p(j,:)); end
            if ~int(l), b(vk)=b(vk)+Ncntrb(gN,p(k,:),p(l,:)); end,  end
        if inGamN(l)
            if ~int(j), b(vl)=b(vl)+Ncntrb(gN,p(l,:),p(j,:)); end
            if ~int(k), b(vl)=b(vl)+Ncntrb(gN,p(l,:),p(k,:)); end,  end
end

uh(~inGamD)=A\b;                               % solve for FE solution
trimesh(t,p(:,1),p(:,2),uh), axis tight      % display

function w=Ncntrb(gN,p,q); % compute Neumann contribution by Simpson's rule
w=norm(p-q)*( feval(gN,p) + 2*feval(gN,(p+q)/2) )/6;
```

Example 1. *For a first verification, I consider a problem for which I know that the exact solution is in $H^2(D)$, namely,*

$$-\triangle u = 0 \quad \text{on } D = [0,1]\times[0,1], \qquad \text{and} \qquad u = x^2 - y^2 \quad \text{on } \Gamma_D = \partial D.$$

In particular, $f \equiv 0$ and $\Gamma_N = \emptyset$ so this is a classical Laplace equation example. The solution on D is, of course, $u(x,y) = x^2 - y^2$.

To use `distmesh2d.m` *and* `poissonDN.m` *on this problem, we choose a mesh with typical feature size of $h_0 = 0.1$:*

```
>> fd=inline('drectangle(p,0,1,0,1)','p');  fGam=inline('-1','p');
>> h0=0.1; [p,t]=distmesh2d(fd,@huniform,h0,[0,0;1,1],[0,0;0,1;1,0;1,1]);
>> f=inline('0','p');  gD=inline('p(:,1).^2-p(:,2).^2','p');
>> [uh,un]=poissonDN(f,gD,f,fd,fGam,h0,p,t);  err=max(abs(uh-gD(p)))
```

The result is an error of `err`$= 5.1\times 10^{-4}$. *In fact, we consider $h_0(k) = 0.5(3/5)^k$ for $k = 0,1,\dots,6$ on the same problem. The picture of convergence is in figure 1. It seems we have $O(h_0^2)$ convergence which is optimal.*

Let us now attempt to prove a theorem on error. Recall that $|f|_1^2 = \int_D |\nabla f|^2 = \sum_{i=1}^2 \int_D |f_{x_i}|^2$ and $|f|_2^2 = \sum_{i,j=1}^2 \int_D |f_{x_i x_j}|^2$ while $\|f\|_{L^2}^2 = \int_D |f|^2$ and $\|f\|_{H^1}^2 = \|f\|_{L_2}^2 + |f|_1^2$. That is, $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^1}$ are norms while $|\cdot|_1$ and $|\cdot|_2$ are semi-norms.

Theorem 1. *Suppose $u \in H^1_{g_D}(\Gamma_D)$ solves the variational problem (2) and that u_h satisfies (3) and (4) for a fixed triangulation $\mathcal{T}$. Let $\gamma = \sum_{j=N+1}^{N_p} g_D(p_j)\varphi_j \in H^1(D)$. Then*

$$|u - u_h|_1 \le |u - (v+\gamma)|_1 \tag{5}$$

for all $v \in V_h$. In particular, suppose $\mathcal{T}$ is a regular *triangulation with $h > 0$ a bound on the maximum side length ("diameter") of triangles $T \in \mathcal{T}$ and with $B > 0$ a bound*

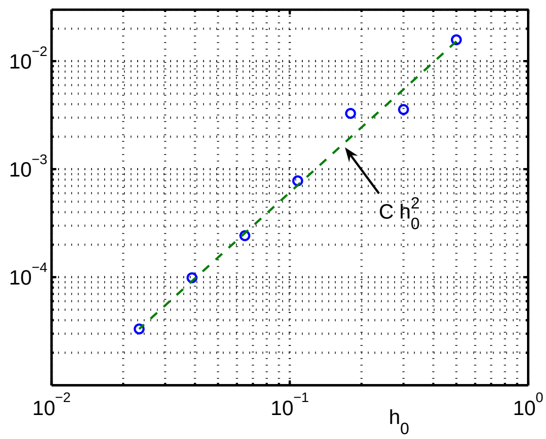

FIGURE 1. Convergence of the `poissonDN.m` on a simple Laplace equation example.

on the ratios $\frac{h_T}{\rho_T}$ *of the diameter* h_T *to the radius* ρ_T *of the largest inscribed circle inside* $T \in \mathcal{T}$*. It follows that*

$$|u - u_h|_1 \le CBh|u|_2 \tag{6}$$

for some C *depending only on* D*.*

Proof. By subtraction of (4) from (2) we have

$$\int_D \nabla(u - u_h) \cdot \nabla v = 0 \qquad \text{for all } v \in V_h. \tag{7}$$

In particular, $\int_D \nabla(u - u_h) \cdot (u_h - \gamma - v) = 0$ if $v \in V_h$. Thus

$$\begin{aligned} |u - u_h|_1^2 &= \int_D \nabla(u - u_h) \cdot \nabla(u - u_h + u_h - \gamma - v) = \int_D \nabla(u - u_h) \cdot \nabla(u - \gamma - v) \\ &\le |u - u_h|_1 \, |u - (v + \gamma)|_1, \end{aligned}$$

by Cauchy-Schwarz. Estimate (5) follows.

We now use the theory of interpolation error in section 4.2 of [5]. Estimate (4.17) applies, namely $|u - \pi_h u|_1 \le CBh|u|_2$, where $\pi_h u$ is the piecewise-linear interpolant of u, and furthermore $\pi_h u = v + \gamma$ for $v \in V_h$. Thus (6) follows from (5). □

To make this result useful we need to know that $|\cdot|_1$ controls $\|\cdot\|_{H^1}$ when we consider functions which are zero on a "significant" part of ∂D.

Lemma 2 (generalization of Poincaré's inequality; proposition (5.3.3) in [1])**.** *Suppose* D *is a union of domains that each are star-shaped with respect to finitely-many discs (see* [1]*), or, for example, suppose* D *is a polygon. Suppose* $\Gamma_D \subset \partial D$ *is closed and that* $\text{meas}(\Gamma_D) > 0$ *(* $\text{length}(\Gamma_D) > 0$ *if* Γ_D *is a countable union of smooth curve segments). There exists* $C > 0$ *depending only on* D *and* Γ_D *such that*

$$\|v\|_{H^1} \le C|v|_1 \qquad \textit{for all} \qquad v \in H_0^1(\Gamma_D).$$

Thus we have the following corollary to theorem 1.

Corollary 3. *Assuming $u, u_h, \mathcal{T}$, and γ are as in theorem 1, and assume D is (for example) a polygon. Suppose $\epsilon \in H^1(D)$ is some function such that $\epsilon\big|_{\Gamma_D} = g_D - \gamma\big|_{\Gamma_D} = u\big|_{\Gamma_D} - u_h\big|_{\Gamma_D}$. Then*

$$\|u - u_h\|_{H^1} \le C_1 B h |u|_2 + C_2 \|\epsilon\|_{H^1}$$

for C_i depending only on D and Γ_D.

Proof. Note $u - u_h - \epsilon \in H^1_0(\Gamma_D)$. Thus from lemma 2 and theorem 1,

$$\begin{aligned}\|u - u_h\|_{H^1} &\le \|u - u_h - \epsilon\|_{H^1} + \|\epsilon\|_{H^1} \le C|u - u_h - \epsilon|_1 + \|\epsilon\|_{H^1} \\ &\le C|u - u_h|_1 + (C+1)\|\epsilon\|_{H^1} \le CC'Bh|u|_2 + (C+1)\|\epsilon\|_{H^1}.\end{aligned}$$

□

In fact ϵ can be chosen small if g_D is well-approximated by piecewise linear functions.

Exercise. **(a)** *Assume g_D is continuous and has bounded second derivative on each segment of Γ_D; recall we assume D is a polygon. Consider a triangle T for which one edge is contained in the Dirichlet boundary Γ_D. In particular, put coordinates on T and suppose T has dimensions δ, d, ω as shown in figure 2; the $x = 0$ edge is in Γ_D. Let $\psi(y) = g_D - \gamma\big|_{\Gamma_D}$ so $\psi(0) = \psi(\delta) = 0$. Define*

$$\epsilon(x, y) = \frac{d - x}{d}\, \psi\left(\frac{dy - \omega x}{d - x}\right).$$

Show that

$$\int_T |\epsilon|^2 \le \|\psi\|_\infty^2 \frac{\delta d}{3} \quad \text{and} \quad \int_T |\nabla \epsilon|^2 \le \|\psi\|_\infty^2 \frac{\delta}{d} + \|\psi'\|_\infty^2 \int_T \left(2\left(\frac{y - \omega}{d - x}\right)^2 + 1\right) dx\, dy.$$

(b) *Show that*

$$-\frac{\omega}{d} \le \frac{y - \omega}{d - x} \le \frac{\delta - \omega}{d}$$

if $(x, y) \in T$. Now assume that the triangulation is regular so that the ratio h_T/ρ_T is bounded by B and so that diameters h_T are bounded by h. Show there exists constants $C_i > 0$ so that

$$\int_T |\epsilon|^2 \le C_1 \|\psi\|_\infty^2 h^2 \quad \text{and} \quad \int_T |\nabla \epsilon|^2 \le C_2 \|\psi\|_\infty^2 + C_3 \|\psi'\|_\infty^2 h^2.$$

(c) *By one-dimensional interpolation theory (*[5, page 25]*) $\|\psi\|_\infty \le \|g_D''\|_\infty \delta^2/8 \le \|g_D''\|_\infty h^2/8$ and $\|\psi'\|_\infty \le \|g_D''\|_\infty \delta \le \|g_D''\|_\infty h$ on each T which meets Γ_D. Now define ϵ on the entire triangulation as zero on each T which does not meet Γ_D. Show that*

$$\|\epsilon\|_{L^2} \le C_4 \|g_D''\|_\infty h^{5/2} \quad \text{and} \quad |\epsilon|_1 \le C_5 \|g_D''\|_\infty h^{3/2}.$$

*(*Hint*: There are $O(h^{-1})$ triangles T along Γ_D.)*

From the above exercise we have another corollary of theorem 1.

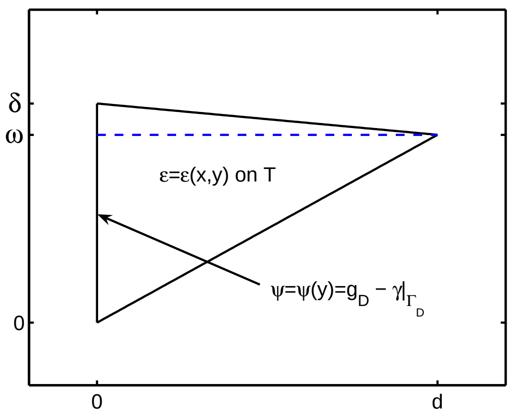

FIGURE 2. A function $\epsilon(x,y)$ can be chosen to be small in the $\|\cdot\|_{H^1}$ sense on a boundary triangle T which meets Γ_D because $\psi = g_D - \gamma\big|_{\Gamma_D}$ is small on the boundary of D if g_D has bounded second derivative.

Corollary 4. *Assuming $u, u_h, \mathcal{T}$, and γ are as in theorem 1, and assume D is a polygon. Suppose g_D is continuous on Γ_D with bounded second derivative on each segment of Γ_D. Then*

$$\|u - u_h\|_{H^1} \le C_1 h |u|_2 + C_2 h^{3/2} \|g_D''\|_\infty$$

for C_i depending only on D, Γ_D, and B.

This result is still unsatisfying, however, because it is possible that $u \notin H^2(D)$ for a boundary value problem of the form (1).

Example 2. *Let $D = \{0 < x < \pi,\ 0 < y < 1\} \subset \mathbb{R}^2$ and consider the problem $\triangle u = 0$, $u(x,0) = u(0,y) = u(\pi,y) = 0$, and $u(x,1) = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \pi/2 \le x \le \pi \end{cases}$. This is an example of problem (1) with D a convex polygon, $f \equiv 0$, $\Gamma_D = \partial D$, $\Gamma_N = \emptyset$, and g_D continuous. The solution, found by separation of variables and Fourier sine series, is*

$$u(x,y) = \sum_{k=1}^{\infty} \frac{4\sin(k\pi/2)}{\pi k^2} \sin(kx) \frac{\sinh(ky)}{\sinh k}. \tag{8}$$

Is $u \in H^2(D)$? A picture of u, which "tents" to a point at $(x,y) = (\pi/2, 1)$, will cause the reader to doubt it. In fact, the following MATLAB *produces figure 3:*

```
>> N=40; y=0:.01:1; x=pi*y; [xx,yy]=meshgrid(x,y);
>> k=1:N; c=4*sin(k*pi/2)./(pi*k.^2.*sinh(k));
>> u=zeros(101); for k=1:N, u=u+c(k)*sin(k*xx).*sinh(k*yy); end
>> mesh(x,y,u), axis tight, xlabel x, ylabel y
```

Indeed, if u^N is the partial sum then

$$\int_D (u^N_{xx})^2 = \frac{8}{\pi} \sum_{k=1}^{N} \frac{\sin^2(k\pi/2)}{\sinh^2 k} \int_0^1 \sinh^2(ky)\,dy = \frac{4}{\pi} \sum_{k=1}^{N} \frac{\sin^2(k\pi/2)}{k} \frac{\sinh(2k) - 2k}{\cosh(2k) - 1}$$

using orthogonality ($\int_0^\pi \sin(jx)\sin(kx)\,dx = \frac{\pi}{2}\delta_{jk}$) *and a few hyperbolic identities. But* $\frac{\sinh(2k)-2k}{\cosh(2k)-1} \to 1$ *as* $k \to \infty$ *and also* $\sin^2(2j\pi/2) = 0$*, so, by Bessel's inequality,*

$$|u|_2^2 \geq \int_D (u_{xx}^N)^2 \geq c \sum_{j=1}^{N} \frac{1}{2j-1},$$

for $c > 0$*. The right-hand sum diverges as* $N \to \infty$*. That is,* $u \notin H^2(D)$ *even though* $u \in C^\infty(D)$*. (Of course,* $u \notin C^\infty(\overline{D})$*.)*

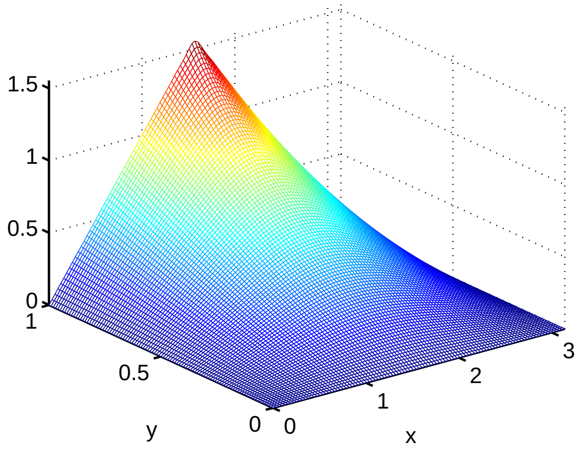

FIGURE 3. A solution $u = u(x, y)$ to Laplace's equation with continuous Dirichlet boundary values for which u is not in $H^2(D)$.

Now we report even worse news. That is, even if all the boundary data is zero, a transition from Dirichlet to Neumann boundary condition can cause a failure of $H^2(D)$ regularity for solutions to Poisson's equation.

Example 3. *Let* D *be the upper half of the unit disc. Suppose* Γ_D *is the union of the part of* ∂D *which is on the positive real axis and the part of* ∂D *which is the upper unit circle. Thus* Γ_N *is the (open) part of* ∂D *which is on the negative real axis. Let*

$$u(r, \theta) = (1 - r^2) r^{1/2} \sin(\theta/2)$$

in polar coordinates. Note $u = 0$ *on* Γ_D *and* $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial \theta} = 0$ *on* Γ_N*. Furthermore*

$$\triangle u = -f(r, \theta) = -6r^{1/2} \sin(\theta/2),$$

that is, u *solves a Poisson equation* $-\triangle u = f$ *where* f *is continuous on* $\overline{D}$ *(and thus in* $L^2(D)$*, of course). But, as the reader can check,* $u_{rr} \approx -\frac{1}{4} r^{-3/2}$ *near* $r = 0$*, and thus* $u_{rr} \notin L^2(D)$*. Thus* $u \notin H^2(D)$*.*

Figure 4 shows this surface. It was generated by

```
>> fd=inline('max(sqrt(sum(p.^2,2))-1,-p(:,2))','p');
>> [p,t]=distmesh2d(fd,@huniform,0.06,[-1,-1;1,1],[-1,0;1,0]);
>> [theta,r]=cart2pol(p(:,1),p(:,2)+1e-8);  u=(1-r.^2).*sqrt(r).*sin(theta/2);
>> trimesh(t,p(:,1),p(:,2),u),  axis tight,  xlabel x,  ylabel y
```

I have used a triangular mesh to show the surface but there was no finite element calculation *of* u*. For more on this example, see* [1, section 5.5]*.*

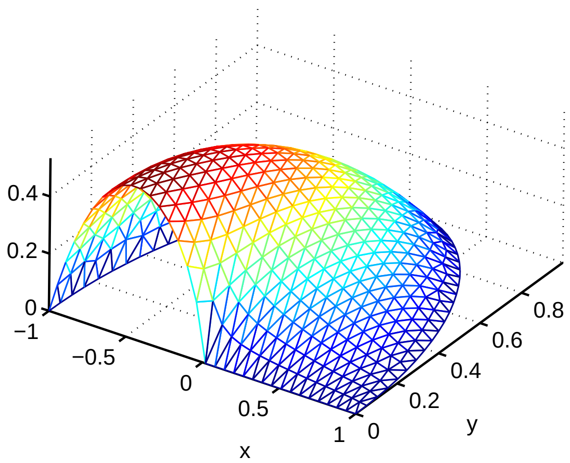

FIGURE 4. A solution $u = u(x, y)$ to Poisson's equation $-\triangle u = f$ with f continuous, and zero Dirichlet and Neumann boundary data, but for which u is not in $H^2(D)$.

Thus warned, we recall the regularity result which does exist for Dirichlet boundary conditions. We use it, along with a *duality argument* [5, section 4.7], to give a final closer-to-optimal rate theorem on convergence of the FEM in the case of nonhomogeneous Dirichlet boundary conditions.

Lemma 5. *(*Elliptic regularity up to the boundary. See [3, section 6.3.2] and [4].*) Suppose* ∂D *is either* C^2 *or convex. Suppose* Γ_D *is either empty or* $\Gamma_D = \partial D$*. There exists* C *depending on* D *and* Γ_D *so that if* w *solves* $-\triangle w = f$*,* $w = 0$ *on* Γ_D*, and* $\frac{\partial w}{\partial n} = 0$ *on* $\Gamma_N = \partial D \setminus \Gamma_D$ *then* $|w|_2 \le C\|f\|_{L^2}$*.*

Theorem 6. *Suppose* D *is a convex polygon and* $\Gamma_D = \partial D$*. Suppose* $\mathcal{T}$ *is a regular triangulation of* D *satisfying the hypotheses given on page 2 and the hypotheses of theorem 1. In particular, assume a bound* B *on ratios* ρ_T/h_T*. There are constants* C_i *depending only on* D*,* Γ_D*, and* B *so that if* u *solves* (2) *and* u_h *solves* (4) *then*

$$\|u - u_h\|_{L^2} \le C_1 h^2 |u|_2 + C_2 h^{3/2} \|g_D''\|_\infty.$$

Remark. The proof below follows the common convention that the letter C is a "generic positive constant" which absorbs positive combinations of previous constants.

Proof. Let $e = u - u_h$ be the error. Consider the "dual problem"

$$-\triangle w = e, \qquad w = 0 \text{ on } \partial D.$$

Elliptic regularity implies $|w|_2 \le C\|e\|_{L^2}$. Also, because $\int_D \varphi e = -\int_D \varphi \triangle w = \int_D \nabla\varphi \cdot \nabla w - \int_{\partial D} \varphi \frac{\partial w}{\partial n} = \int_D \nabla\varphi \cdot \nabla w$ for all $\varphi \in H_0^1(D)$, $|w|_1^2 = \int_D we \le \|w\|_2\|e\|_{L^2} \le \|w\|_{H^1}\|e\|_{L^2}$. By lemma 2 it follows that $C\|w\|_{H^1}^2 \le \|w\|_{H^1}\|e\|_{L^2}$, so

$$\|w\|_{H^1} \le C\|e\|_{L^2}. \tag{9}$$

On the other hand,

$$\begin{aligned}\int_D e^2 &= -\int_D e\triangle w = \int_D \nabla e \cdot \nabla w - \int_{\partial D} e\frac{\partial w}{\partial n} = \int_D \nabla e \cdot \nabla(w - \pi_h w) - \int_{\partial D} \epsilon\frac{\partial w}{\partial n} \\ &= \int_D \nabla e \cdot \nabla(w - \pi_h w) - \int_D \nabla\epsilon \cdot \nabla w - \int_D \epsilon\triangle w.\end{aligned}$$

We denote the piecewise linear interpolant of w by $\pi_h w$. We have used equation (7) and we introduce $\epsilon \in H^1(D)$, found in part **(c)** of the previous exercise, which satisfies $\epsilon\big|_{\partial D} = (u - u_h)\big|_{\partial D} = g_D - \gamma\big|_{\partial D}$. It follows that

$$\|e\|_{L^2}^2 \le |e|_1|w - \pi_h w|_1 + |\epsilon|_1|w|_1 + \|\epsilon\|_{L^2}\|e\|_{L^2}. \tag{10}$$

Now we estimate the first two terms on the right of (10) by interpolation theory, theorem 1, elliptic regularity, and equation (9):

$$\begin{aligned}\|e\|_{L^2}^2 &\le |e|_1 Ch|w|_2 + |\epsilon|_1\|w\|_{H^1} + \|\epsilon\|_{L^2}\|e\|_{L^2} \le Ch|u|_2 Ch\|e\|_{L^2} + |\epsilon|_1\|w\|_{H^1} + \|\epsilon\|_{L^2}\|e\|_{L^2} \\ &\le Ch^2|u|_2\|e\|_{L^2} + C|\epsilon|_1\|e\|_{L^2} + \|\epsilon\|_{L^2}\|e\|_{L^2}.\end{aligned}$$

Thus $\|u - u_h\|_{L^2} \le Ch^2|u|_2 + C|\epsilon|_1 + \|\epsilon\|_{L^2}$. From the exercise,

$$\|u - u_h\|_{L^2} \le Ch^2|u|_2 + Ch^{3/2}\|g_D''\|_\infty + Ch^{5/2}\|g_D''\|_\infty \le C_1h^2|u|_2 + C_2h^{3/2}\|g_D''\|_\infty.$$

□

This theorem substantially, but not completely, explains the convergence observed in example 1 and figure 1.

References

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