APPROXIMATIONS OF THE DOMINATION NUMBER OF A GRAPH

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ABSTRACT. Let G be a graph with an ordered set of vertices and maximum degree Δ. The domination number $\gamma(G)$ of G is the minimum order of a set S of vertices such that each vertex not in S is adjacent to some vertex in S. Equivalently, we can label the vertices from $\{0, 1\}$ so that the sum over each closed neighborhood is at least one; the minimum value of the sum of all labels, with this restriction, is the domination number. The fractional domination number $\gamma^*(G)$ is defined in the same way except that the vertex labels are chosen from $[0, 1]$. Let $\gamma_g(G)$ be the approximation of the domination number by the standard greedy algorithm. Computing the domination number is NP-complete; however, we can bound γ by these two more easily computed parameters:

$$\gamma^*(G) \leq \gamma(G) \leq \gamma_g(G).$$

How good are these approximations? Using techniques from the theory of hypergraphs, one can show that, for every graph G of order n,

$$\frac{\gamma_g(G)}{\gamma^*(G)} = O(\log n).$$

On the other hand, we provide examples of graphs for which $\gamma/\gamma^* = \Theta(\log n)$ and graphs for which $\gamma_g/\gamma = \Theta(\log n)$. Lastly, we use our examples to compare two bounds on γ_g.

In the following, G will represent a finite, simple, undirected graph. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degree of G, respectively. We use $N[v]$ to denote the closed neighborhood of a vertex v. The closed neighborhood of a sequence of vertices, e.g., $N[v_1, v_2, \ldots, v_k]$, is the union of the closed neighborhoods of the vertices in the sequence. We denote the domination number of G by $\gamma(G)$. See [9] for an introduction to domination in graphs and definitions of graph-theoretic terms.

We may consider a dominating set as a $0, 1$-weighting of the vertex set so that, in each closed neighborhood, the sum of the weights is at least one. Relaxing the requirement that the weights be integers, we obtain a fractional version of the domination number. Suppose we assign weight $f(v) \in [0, 1]$ to each vertex v. The function $f \colon V(G) \to [0, 1]$ is a *fractional domination* if for each vertex v,

$$\sum_{u \in N[v]} f(u) \geq 1.$$

The *fractional domination number* $\gamma^*(G)$ of G is the minimum sum of the vertex weights, taken over all fractional dominations of G.

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A useful bound is the following, which was discovered independently and appears in [4, 8].

Lemma 1. *For a graph G of order n,*

$$\frac{n}{1+\Delta(G)} \leq \gamma^*(G) \leq \frac{n}{1+\delta(G)}. \quad \square$$

Throughout this paper, we will implicitly assume an ordering on the vertex set of a graph. Given such an ordering, we can approximate the domination number using a greedy algorithm, as follows. Iteratively select vertices $x_1, x_2, \ldots, x_m$ so that, for each $k = 1, 2, \ldots, m$, vertex x_k is chosen so that it dominates as many vertices of $V(G) - N[x_1, x_2, ..., x_{k-1}]$ (that is, not-yet-dominated vertices) as possible. Resolve ties by choosing x_k as early as possible in the ordering on $V(G)$. Stop the iterative process when every vertex is dominated by one of the x_k's. We refer to $x_1, x_2, \ldots, x_m$ as the *greedy dominating sequence.* The *greedy domination number* $\gamma_g(G) = m$ is the number of vertices in this sequence.

Determining the domination number of a general graph is known to be NP-complete (see [7]); it is natural to seek more easily computed approximations. The values of γ^* and γ_g can be determined in polynomial time. Further, the fact that γ lies in the interval $[\gamma^*, \gamma_g]$ follows easily from definitions.

Observation 2. *For every graph G,*

$$\gamma^*(G) \leq \gamma(G) \leq \gamma_g(G). \quad \square$$

We study the relationships of these three parameters further.

Techniques from the theory of hypergraphs can be used to show that the ratio $\gamma_g(G)/\gamma^*(G)$ is $O(\log \Delta)$, and thus $O(\log n)$, where n is the order of G; see Theorem 4, below. Thus $\gamma(G)$ must lie within a relatively small interval. We produce examples showing that, asymptotically, we can do no better. We show that $\gamma(G)/\gamma^*(G)$ can be $\Theta(\log n)$, and then we show that $\gamma_g(G)/\gamma(G)$ can be $\Theta(\log n)$.

Since γ_g is a useful upper bound on γ, it is worthwhile to consider upper bounds on γ_g. One such bound follows immediately from the above discussion:

$$\gamma_g(G) \leq c\gamma^*(G) \log n,$$

for some constant c, where n is the order of G. Another class of bounds are those in which γ_g is bounded above by a constant multiple of $n \log \delta/\delta$. The first of these is found in [1] (see their Theorem 2.2 and the remarks following it). A slightly improved bound is given in [3, Thm. 2]; we state this below.

Theorem 3 (Clark, Shekhtman, Suen, and Fisher [3])**.** *For every graph G of order n,*

$$\gamma_g(G) \leq n \left[1 - \prod_{i=1}^{\delta+1} \frac{i\delta}{i\delta + 1} \right],$$

where $\delta = \delta(G)$. $\square$

We note that the right side of the above inequality is $\Theta(n \log \delta/\delta)$. We will compare these two bounds on γ_g, using examples to show that sometimes one is tighter, and sometimes the other is.

In the following result, we will use a concept dual to fractional domination. A function $f\colon V(G)\to[0,1]$ is a *fractional packing* if for each vertex v,

$$\sum_{u\in N[v]} f(u)\le 1.$$

Note that the maximum total weight of $V(G)$, taken over all fractional packings, and the minimum total weight of $V(G)$, taken over all fractional dominations, are described by dual linear programs (see [9, Chapter 4] or [4, Section 3]). Thus, by the principle of strong duality, given a fractional packing on a graph G, the total weight of the vertex set is at most $\gamma^*(G)$.

We now prove an upper bound on $\gamma_g(G)/\gamma^*(G)$. This is a special case of a more general result on vertex covers of hypergraphs and is similar to a bound found in [10, 11] (see also [14, Thm. 77.2]).

Theorem 4. *For every graph G,*

$$\frac{\gamma_g(G)}{\gamma^*(G)}\le 1+\ln\bigl[1+\Delta(G)\bigr].$$

Proof. Set $m=\gamma_g(G)$. Let $x_1,x_2,\dots,x_m$ be the greedy dominating sequence. For each vertex v of G, let $g(v)$ be the first vertex in the greedy dominating sequence that dominates v. Let $F(v)$ be the set of all vertices of G that are first dominated by $g(v)$; that is, $F(v)=N[x_k]-N[x_1,x_2,\dots,x_{k-1}]$, where $x_k=g(v)$. Let $w(v)=\frac{1}{|F(v)|}$. So $w(v)$ is the reciprocal of the number of vertices that are dominated in the same step of the greedy algorithm as v. Note that $\sum_{u\in F(v)} w(u)=1$, and thus $\sum_{v\in V(G)} w(v)=m$.

Our proof is based on that of [14, Thm. 77.2], and proceeds as follows. We assign weight $w(v)$ to each vertex v. We find upper bounds on the weights of vertices lying in a closed neighborhood, and conclude that, if each vertex v is given weight $w(v)/\bigl(1+\ln\bigl[1+\Delta(G)\bigr]\bigr)$, then the result is a fractional packing. Applying linear programming duality, we then obtain a lower bound on $\gamma^*(G)$, from which our result follows.

Let v be a vertex of G. We list the elements of $N[v]$ in the order in which they were dominated in the greedy algorithm. Letting $p=1+\deg(v)$, we represent $N[v]$ as $\{u_1,u_2,\dots,u_p\}$, where, if $g(u_i)$ comes before $g(u_j)$ in the greedy dominating sequence, then $i<j$.

We claim that $w(u_i)\le\frac{1}{p+1-i}$ for each u_i. Suppose that $|F(u_i)|<p+1-i$, for some u_i. Then $|F(u_i)|<|\{u_i,u_{i+1},\dots,u_p\}|$, and so replacing $g(u_i)$ by v in the greedy dominating sequence would increase the number of vertices dominated at this step in the greedy algorithm. However, this contradicts the definition of greedy dominating sequence, and so $|F(u_i)|\ge p+1-i$. Thus,

$$w(u_i)=\frac{1}{|F(u_i)|}\le\frac{1}{p+1-i},$$

as claimed.

Hence, for each vertex v we have,

$$\sum_{u\in N[v]} w(u)\le\sum_{i=1}^{p}\frac{1}{p+1-i}=\sum_{i=1}^{p}\frac{1}{i}\le 1+\ln p\le 1+\ln\bigl[1+\Delta(G)\bigr],$$

Dividing by $1 + \ln\bigl[1 + \Delta(G)\bigr]$, we obtain

$$\sum_{u \in N[v]} \frac{w(u)}{1 + \ln\left[1 + \Delta(G)\right]} \le 1,$$

and so assigning weight $w(v)/\bigl(1 + \ln\bigl[1 + \Delta(G)\bigr]\bigr)$ to each vertex v, results in a fractional packing. Therefore, as noted before the statement of the theorem, the sum of all vertex weights is bounded above by $\gamma^*(G)$. That is,

$$\sum_{v \in V(G)} \frac{w(v)}{1 + \ln\left[1 + \Delta(G)\right]} \le \gamma^*(G).$$

Multiplying by $1 + \ln\bigl[1 + \Delta(G)\bigr]$, we obtain

$$\gamma_g(G) = m = \sum_{v \in V(G)} w(v) \le \bigl(1 + \ln\bigl[1 + \Delta(G)\bigr]\bigr)\,\gamma^*(G).$$

Dividing by $\gamma^*(G)$ yields our result. □

Hence the following.

Corollary 5. *For any graph G of order n with maximum degree $\Delta \ge 2$*

$$\gamma(G) \le c_1 \ln(\Delta)\gamma^*(G)$$

and

$$\gamma(G) \le c_2 \ln(n)\gamma^*(G),$$

where c_1 and c_2 are appropriately chosen constants. □

The preceding theorem and corollary place restrictions on the value of γ. We now show that these restrictions are asymptotically best possible up to a constant factor. We begin with a construction of a family of graphs in which γ lies near the high end of the interval $[\gamma^*, \gamma_g]$. Later, we will obtain better results using random graphs.

Example 6. Given a positive integer t, we construct a graph J_t of order $n = (2t)^{2t-1}$ so that

$$\gamma(J_t) = 2t = \Theta\left(\frac{\log n}{\log \log n}\right),$$

and

$$\gamma^*(J_t) = e + o(1) = \Theta(1).$$

Let t be a positive integer. Set $d = 2t - 1$ and $n = (2t)^d$. Let G be the graph $K_{2t} - tK_2$ (that is, K_{2t} with a matching removed). Let J_t be the graph whose vertices are d-tuples of the form $(x_1, x_2, \ldots, x_d)$ where each x_i is a vertex in G. Let vertices $(x_1, x_2, \ldots, x_d)$ and $(y_1, y_2, \ldots, y_d)$ be adjacent in J_t if for each i, the vertices x_i and y_i are equal or adjacent in G. (The way in which J_t is constructed from G is often called the "strong [direct] product".) We note that J_t has order n.

We show that J_t has the required properties. For each vertex v of G, denote by $\overline{v}$ the unique vertex in G that is not adjacent to v.

Let S be a set of d vertices of J_t. We write $S = \bigl\{\bigl(x_1^i, x_2^i, \ldots, x_d^i\bigr) \mid i = 1, 2, \ldots, d\bigr\}$. Let $u = \Bigl(\overline{x_1^1}, \overline{x_2^2}, \ldots, \overline{x_d^d}\Bigr)$. Then u is not adjacent to any vertex in S, and so S is not a dominating set. Hence, the domination number of J_t is at least $d + 1$. Now let A be the set of all vertices in J_t of the form $(v, v, v, \ldots, v)$ where v is a vertex in G. Since there are $d+1$ such vertices, but only d coordinates, every vertex of J_t must be dominated by at least one vertex of A. Thus, A is a dominating set of size $d+1$, and so $\gamma(J) = d+1 = 2t$.

Note that J_t is regular of degree $(2t-1)^d - 1$. By Lemma 1,

$$\gamma^*(J_t) = \frac{n}{(2t-1)^d} = \frac{(d+1)^d}{d^d} = e + o(1). \quad \square$$

For the graph J_t of Example 6, $\gamma/\gamma^* = \Theta(\log n/\log\log n)$. This ratio is not as high as we would like. Better examples are provided by random graphs, for which γ/γ^* is, with high probability, $\Theta(\log n)$.

Given a natural number n, let R_n be a random graph on n labeled vertices with edge probability 1/2. Given a graphical property P we say that R_n *almost surely* (a.s.) has P if the probability that R_n has P goes to one as n approaches infinity. See [13] for an introduction to random graphs.

It is known (see [5, 15, 16]) that the domination number of R_n is almost surely $\Theta(\log n)$. We give a short proof below.

Theorem 7. *Almost surely,*

$$\gamma^*(R_n) = 2 + o(1)$$

and

$$\gamma(R_n) = \log_2 n + o(\log n). \quad \square$$

Proof. From [6] we know a.s.

$$\big(1 - o(1)\big)\frac{n}{2} \le \delta(R_n) \le \Delta(R_n) \le \big(1 + o(1)\big)\frac{n}{2}.$$

Applying Lemma 1 we see that a.s. $\gamma^*(R_n) = 2 + o(1)$.

From [2, 12] we know the independence number of R_n is a.s. $\log_2 n + o(\log n)$. Hence, a.s. $\gamma(R_n) \le \log_2 n + o(\log n)$. Fix ϵ so that $0 < \epsilon < 1$. Set $p = \lfloor (1-\epsilon)\log_2 n \rfloor$. Let S be a subset of $V(G)$ with order p. If v is a vertex not in S then the probability that S dominates v is $1 - \left(\frac{1}{2}\right)^p$. Hence, the probability that S dominates R_n is $\left[1 - \left(\frac{1}{2}\right)^p\right]^{n-p}$. Let E be the expected number of p-sets that dominate R_n. Then,

$$\begin{aligned} E = \binom{n}{p}\left[1 - \left(\frac{1}{2}\right)^p\right]^{n-p} &\le n^p e^{-(1/2)^p(n-p)} \\ &\le e^{p\ln(n) - \left(n/n^{1-\epsilon}\right)} e^{p/n^{1-\epsilon}} \\ &\le c e^{p\ln(n) - n^{\epsilon}}, \end{aligned}$$

for some constant c. But the last expression goes to zero. Hence, R_n a.s. has no dominating p-set. This leads to the desired result. $\square$

Letting $G_n = R_n$, we obtain the following.

Corollary 8. *There exist graphs G_n, for infinitely many integers n, so that each G_n has order n, and*

$$\frac{\gamma(G_n)}{\gamma^*(G_n)} = \Theta(\log n). \quad \square$$

Thus, the bounds in Corollary 5 are asymptotically best possible. We have proven this using probabilistic methods; we ask whether an explicit construction can be found.

Problem 9. *Find an explicit construction of graphs G_n, for infinitely many integers n, so that each G_n has order n, and*

$$\frac{\gamma(G_n)}{\gamma^*(G_n)} = \Theta\left(\log n\right). \quad \square$$

We have seen that γ_g/γ^* is $O(\log n)$, and that the ratio γ/γ^* may be $\Theta(\log n)$. In our next example the ratio γ_g/γ is $\Theta(\log n)$. Thus, γ is near the low end of the interval $[\gamma^*, \gamma_g]$, and the greedy algorithm approximates the domination number relatively poorly.

Example 10. Given an integer $t \geq 4$, we construct a graph H_t of order $n = 2^{t+2}$ so that

$$\gamma^*(H_t) = \gamma(H_t) = 4$$

and

$$\gamma_g(H_t) = t.$$

Let $t \geq 4$ be a natural number. Let u_1, u_2, u_3, u_4 be vertices and set $S = \{u_1, u_2, u_3, u_4\}$. To construct H_t, begin with the union of S and t disjoint cliques:

$$S \cup \left[K_4 \cup K_8 \cup K_{16} \cup \cdots \cup K_{2\cdot 2^t}\right].$$

Add additional edges so that each vertex of S is adjacent to one quarter of the vertices in each clique, and no two vertices of S have any common neighbors. Let H_t be the resulting graph. We note that the order of H_t is

$$4 + 4\left[1 + 2 + 4 + \cdots + 2^{t-1}\right] = 2^{t+2}.$$

If we approximate $\gamma(H_t)$ with the greedy algorithm, we will never choose any vertex in S. The greedy dominating sequence will contain one vertex from each of the cliques used to construct H_t. Since $t \geq 4$ the first four such vertices chosen will dominate the four vertices in S, and so $\gamma_g(H_t) = t$.

Given a fractional domination of H_t, the total weight of the vertices in each $N[u_i]$ is at least 1. Since the sets $N[u_1]$, $N[u_2]$, $N[u_3]$, $N[u_4]$ are disjoint, we have $\gamma^*(H_t) \geq 4$. On the other hand, S dominates H_t, and so $\gamma(H_t) \leq 4$. Thus,

$$4 \leq \gamma^*(H_t) \leq \gamma(H_t) \leq 4,$$

and so $\gamma^*(H_t) = \gamma(H_t) = 4$. $\square$

Letting $n = 2^{t+2}$, and letting G_n be H_t from the above example, we obtain the following.

Corollary 11. *There exist graphs G_n, for infinitely many integers n, so that each G_n has order n, and*

$$\frac{\gamma_g(G_n)}{\gamma(G_n)} = \Theta(\log n). \quad \square$$

We now consider upper bounds on γ_g. By Theorem 4 we have, for a graph G of order n,

$$\gamma_g(G) \leq c_1 \gamma^*(G) \log n, \tag{1}$$

for some constant c_1. And by Theorem 3, we have

$$\gamma_g(G) \le c_2 \frac{n \log \delta(G)}{\delta(G)}, \tag{2}$$

for some constant c_2.

Consider these bounds for the graph H_t from Example 10. We have $\gamma^*(H_t) = 4$, and clearly $\delta(H_t) = 4$. Thus, letting n be the order of H_t, the right-hand side of (1) is $\Theta(\log n)$, while the right-hand side of (2) is $\Theta(n)$, making (1) by far the tighter bound.

On the other hand, let t be a positive integer, and let G be a t-clique with a pendant vertex joined to each clique vertex (a "hairy clique"). Letting n be the order of G, we have $\gamma^*(G) = t = n/2$, and $\delta(G) = 1$. Thus, the right-hand side of (1) is $\Theta(n \log n)$, while the right-hand side of (2) is $\Theta(n)$, making (2) the tighter bound.

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